MATH 520 Axiomatic Systems and their Properties

"Drafted by Thomas Jefferson between June 11 and June 28, 1776, the Declaration of Independence is at once the nation's most cherished symbol of liberty and Jefferson's most enduring monument. Here, in exalted and unforgettable phrases, Jefferson expressed the convictions in the minds and hearts of the American people. The political philosophy of the Declaration was not new; its ideals of individual liberty had already been expressed by John Locke and the Continental philosophers. What Jefferson did was to summarize this philosophy in "self-evident truths" and set forth a list of grievances against the King in order to justify before the world the breaking of ties between the colonies and the mother country."
(http://www.archives.gov/national-archives-experience/charters/declaration.html)

The second paragraph of the Delaration of Independence begins with:

> We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.

These are the "axioms" of Jeffersons philosophy. They were rejected by King George III of England and thus the Revolutionay War began.

The Axiomatic Method

1. Any axiomatic system must contain a set of technical terms that are deliberately chosen as undefined terms (*primitives*) and are subject to the interpretation of the reader.

2. All other technical terms of the system are ultimately defined by means of the undefined terms. These terms are the definitions of the system.

3. The axiomatic system contains a set of statements, dealing with undefined terms and definitions, that are chosen to remain unproved. These are the axioms (*postulates*) of the system.

4. All other statements of the system must be logical consequences of the axioms. These derived statements are called the theorems of the axiomatic system.

If an *interpretation* of a system (this results from giving each undefined term in a system a particular meaning) leads to all the axioms being correct the interpretation is called a *model*. Models can be *concrete* or *abstract*.

Models that are essentially the same are said to be *isomorphic* and the one-to-one correspondence between models is called *isomorphism*.

Properties of Axiomatic Systems

1) Consistency – Impossible to deduce from the axioms a new theorem that contradicts any axiom or previously proved theorem. Concrete models establish *absolute consistency*.

2) Independence – Each axiom cannot be logically deduced from the other axioms in the system.

3) Completeness – Impossible to add an additional consistent and independent axiom without adding additional undefined terms.

Some Finite Geometries

1) Four-Point Geometry

Undefined Terms: *point, line, on*

Axiom 1. There exist exactly four points. (This is an *existence* axiom)

Axiom 2. Any two distinct points have exactly one line on both of them. (This is an *incidence* axiom)

Axiom 3. Each line is on exactly two points. (This is an *incidence* axiom)

Definition: Two lines on the same point are said to intersect and are called intersecting lines.

Definition: Two lines that do not intersect are called parallel lines.

Concrete models of Four-Point Geometry:

Incidence Matrix for Four-Point Geometry:

	A	B	C	D
l	1	1	0	0
m	0	1	1	0
n	0	0	1	1
o	1	0	0	1
p	1	0	1	0
q	0	1	0	1

2) Four-Line Geometry: This geometry is the *plane dual* of four-point geometry. In each axiom the words point and line have been interchanged.

Undefined Terms: *point, line, on*

Axiom 1. There exist exactly four lines. (This is an *existence* axiom)

Axiom 2. Any two distinct lines have exactly one point on both of them. (This is an *incidence* axiom)

Axiom 3. Each point is on exactly two lines. (This is an *incidence* axiom)

Concrete models of Four-Line Geometry:

Incidence Matrix for Four-Line Geometry:

l_1	l_2	l_3	l_4
A	A	B	E
B	E	F	F
C	D	D	C

3) Fano's Geometry: Named after Italian mathematician Gino Fano (1871- 1952). In 1892, Fano considered a finite 3-dimensional geometry consisting of 15 points, 35 lines, and 15 planes. One such plane yields this geometry.

Undefined Terms: *point, line, on*

Axiom 1. There exists at least one line. (This is an *existence* axiom)

Axiom 2. There are exactly three points on every line. (This is an *incidence* axiom)

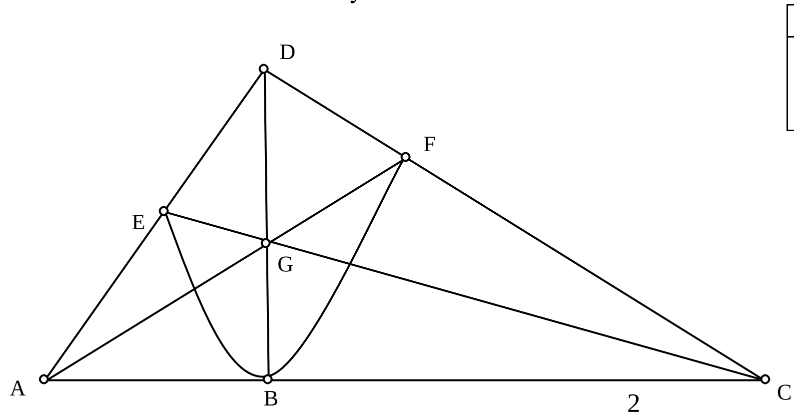
Axiom 3. Not all points are on the same line. (This is an *incidence* axiom)

Axiom 4. There is exactly one line on any two distinct points. (This is an *incidence* axiom)

Axiom 5. There is at least one point on any two distinct lines. (This is an *incidence* axiom)

Concrete models of Fano's Geometry:

l_1	l_2	l_3	l_4	l_5	l_6	l_7
A	A	A	B	C	C	E
B	G	E	G	G	F	B
C	F	D	D	E	D	F



4) Young's Geometry: This geometry uses the first four axioms of Fano with a different Axiom 5

Undefined Terms: *point, line, on*
Axiom 1. There exists at least one line. (*existence)*
Axiom 2. There are exactly three points on every line. (*incidence)*
Axiom 3. Not all points are on the same line. (*incidence)*
Axiom 4. There is exactly one line on any two distinct points. (*incidence)*
Axiom 5. For each line *l* and each point *P* not on *l*, there exists exactly one line on *P* that does not contain any points on *l*. (*incidence)*

A concrete model of Young's Geometry:

Theorems to be Proven:

Four-Point Theorem 1. In the four-point geometry, if two distinct lines intersect, then they have exactly one point in common.
Four-Point Theorem 2. The four-point geometry has exactly six lines.
Four-Point Theorem 3. Each point of the four-point geometry has exactly three lines on it.
Four-Point Theorem 4. In the four-point geometry, each distinct line has exactly one line parallel to it.

Four-Line Theorem 1. The four-line geometry has exactly six points.
Four-Line Theorem 2. Each line of the four-line geometry has exactly three points on it.
Four-Line Theorem 3. A set of two lines cannot contain all the points of the geometry.
Four-Line Theorem 4. There exists a pair of points in the geometry not joined by a line.

Fano's Theorem 1. In Fano's geometry, two distinct lines have exactly one point in common.
Fano's Theorem 2. Fano's geometry contains exactly seven points and seven lines.
Fano's Theorem 3. Each point is on exactly three lines.
Fano's Theorem 4. The set of lines on any point contains all the points of the geometry.

Young's Theorem 1. Every point in Young's geometry is on at least four lines.
Corollary 1. Every point in Young's geometry is on exactly four lines.
Lemma 1. If lines *l* and *m* intersect at a point *P*, and if line *l* is parallel to line *n*, then lines *m* and *n* share some common point *Q*.
Lemma 2. Two lines parallel to the same line are parallel to each other.
Young's Theorem 2. Every line has exactly two lines that are parallel to it
Young's Theorem 3. Young's geometry contains exactly 12 lines.
Young's Theorem 4. Young's geometry contains exactly 9 points.