Laboratory 2
MATH 301 Honors

Rules: Answer the questions below. Please use a separate side of a piece of paper for each question. You may consult your text and use *Mathematica* (**but no other sources animate or inanimate**) for this test. You must show your work. If you use *Mathematica* to make a calculation, you must show the set up that leads to the calculation. Please direct any questions you may have to your instructor.

1. ***The TNB frame and Torsion*** This set of exercises will develop the idea of *torsion*, which together with curvature can be used to understand a curve in three dimensions. It may be helpful to recall the following definitions, noting that s is the arc length parameter and that $\mathbf{r}$ is a vector valued function.

- $\mathbf{T}(s) = \mathbf{r}'(s); \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$
- $\mathbf{N}(s) = \dfrac{\mathbf{r}''(s)}{||\mathbf{r}''(s)||}; \mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$
- $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s); \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- $\kappa(s) = ||\mathbf{T}'(s)|| = ||\mathbf{r}''(s)||; \kappa(t) = \dfrac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$

a) Show that $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$.

b) By considering the length of $\mathbf{B}(s)$, show that $\mathbf{B}'(s)$ is perpendicular to $\mathbf{B}(s)$.

c) By differentiating $\mathbf{B}(s)\cdot\mathbf{T}(s)$, show that $\mathbf{B}'(s)$ is perpendicular to $\mathbf{T}(s)$.

d) Use the results of b) and c) to show that $\mathbf{B}'(s)$ is a scalar multiple of $\mathbf{N}(s)$

The negative of the scalar found in d) is called the **torsion** $\tau(s)$ of the curve $\mathbf{r}(s)$. That is,

$$\mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$$

e) By differentiating $\mathbf{N}(s) = \mathbf{B}(s) \times \mathbf{T}(s)$, show that $\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s)$.

The three formulas you have derived

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s), \mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s), \mathbf{B}'(s) = \tau(s)\mathbf{N}(s)$$

are called the **Frenet-Serret formulas**. This system of differential equations can be solved together with an initial condition $\mathbf{r}'(0)$ to recover a unique $\mathbf{r}$. Thus the curvature and torsion functions determine a curve in three dimensions up to the value of $\mathbf{r}'(0)$. We now proceed to find a formula for $\tau(t)$. Recall that $s'(t) = ||\mathbf{r}'(t)||$. In what follows all functions are implicitly function of t.

f) Use the first two Frenet-Serret formulas and the chain rule to show that $\mathbf{T}' = \kappa s'\mathbf{N}$ and $\mathbf{N}' = -\kappa s'\mathbf{T}$.

g) Show that $\mathbf{r}'(t) = s'\mathbf{T}$ and $\mathbf{r}''(t) = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$.

h) Use the results from f) and g) to show that

$$\mathbf{r}'''(t) = \left(s''' - \kappa^2(s')^3\right)\mathbf{T} + \left(3\kappa s's'' + \kappa'(s')^2\right)\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$$

i) Use the results from g) and h) to show that

$$\tau(t) = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||^2}$$

j) Find the curvature and torsion of the circular helix $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$. How do the curvature and torsion change as a and c change?

2. ***Projectile Motion with Air Resistance*** The projectile motion model studied in class did not allow for air resistance; the only force on the projectile was gravity. The force on the ball at any time t was $\mathbf{F} = -mg\mathbf{j}$ where g is the gravitational constant. Newton's second law of motion says that $\mathbf{F} = m\mathbf{a}(t)$, where $\mathbf{a}(t)$ is the acceleration vector of the object at time t. So $\mathbf{a}(t) = -mg\mathbf{j}$. If we write the path of the projectile as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the acceleration vector is $\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$, and our problem was to solve the set of differential equations

$$x''(t) = 0, y''(t) = -g \tag{1}$$

subject to the initial conditions $x'(0) = v_0\cos\alpha$, $y'(0) = v_0\sin\alpha$, $x(0) = 0$, $y(0) = h$, where v_0 is the initial speed of the projectile, α is the initial angle of the projectile, and h was the initial height of the projectile. We found that

$$x(t) = (v_0\cos\alpha)t \text{ and } y(t) = h + (v_0\sin\alpha)t - \frac{1}{2}gt^2 \tag{2}$$

a) Consider a batter hitting a baseball during a typical major league baseball game. We would like to find its range; that is, how far from where the ball is struck that it lands. Use (2) to find an expression for the range of the ball in terms of v_0, g, h, and α.

b) For a batted ball in a major league game, the following values of the parameters are reasonable: $g = 9.8\ \mathrm{m/s^2}$, $v_0 = 50\ \mathrm{m/s}$, $h = 1\ \mathrm{m}$, and $\alpha = 35^\circ = 35\pi/180$ radians. Use your answer to a) to find the range of the ball in this example.

c) The Mathematica notebook accompanying this lab contains code that will allow you to solve the above initial value problem numerically, to find the range of the ball, and to plot its path. Do this and confirm that the range is equal to that you got by solving the problem by hand.

d) Note that the range is given in meters. Does your answer in a) seem reasonable? You may need to investigate how far baseballs usually fly.

e) A more realistic model includes air resistance, but this will add another force on the ball besides the force of gravity. The extra force due to air resistance has direction opposite to the **velocity** of the object and magnitude proportional to the square of the **speed** of the object. That is, if $\mathbf{F}_R$ is the force due to air resistance, $\mathbf{F}_R$ has direction $-\mathbf{v}(t)$ and magnitude $D||\mathbf{v}(t)||^2$, where D is a constant of proportionality. Show that we can express $\mathbf{F}_R$ as

$$\mathbf{F}_R = -D\sqrt{x'(t)^2 + y'(t)^2}x'(t)\mathbf{i} - D\sqrt{x'(t)^2 + y'(t)^2}y'(t)\mathbf{j}$$

and the total force on the ball $\mathbf{F}$ as

$$\mathbf{F} = -D\sqrt{x'(t)^2 + y'(t)^2}x'(t)\mathbf{i} + (-mg - D\sqrt{x'(t)^2 + y'(t)^2}y'(t))\mathbf{j}$$

f) It can be shown that $D = \dfrac{\rho C A}{2}$, where ρ is the density of the air, A is the cross-sectional area of the projectile, and C is called the **drag coefficient**, which depends on the shape of the projectile. Find a new set of differential equations that if solved would give the position of a projectile assuming air resistance is included.

g) The radius of the baseball is 0.0366 m, $C = .5$, $\rho = 1.174$ kg/m^3, and the mass of the baseball is $m = .145$ kg. Using the same initial conditions as in b), edit the commands in the Mathematica notebook accompanying this lab to solve your initial value problem numerically, find the range of the ball, and plot its path. How do your results differ in this case? Are they more realistic?

h) Changes in humidity and altitude will change the density of the air, and thus the range of the baseball. The value of ρ above is for sea level altitude and 25% relative humidity. If the humidity increased to 75%, the new value of ρ would be $\rho = 1.169$, so humid air is actually less dense. How would this change in humidity change the range of the baseball? If the game were played in Denver (altitude 1609 meters), at 25% relative humidity, the new value of ρ would be $\rho = 0.966$, so air becomes less dense with altitude. How would this change in altitude change the range of the baseball?