Laboratory 4
MATH 202 Honors

Rules: Answer the questions below. Please use a separate side of a piece of paper for each question. You may consult your text and use *Mathematica* (**but no other sources animate or inanimate**) for this test. You must show your work. If you use *Mathematica* to make a calculation, you must show the set up that leads to the calculation. Please direct any questions you may have to your instructor. The assignment is due at 3:00 p.m. on Monday, April 15.

1. This exercise will create a fairly good approximation for π.

   a) We know that

   $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \text{ and } \tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

   Use the first identity to show that for $xy \neq -1$,

   $$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$$

   b) Show that

   $$\tan^{-1} \frac{120}{119} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$$

   c) Use the second identity to show that

   $$4\tan^{-1} \frac{1}{5} = \tan^{-1} \frac{120}{119}$$

   and thus deduce the following formula of John Machin (1680-1751):

   $$4\tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239} = \frac{\pi}{4}$$

   d) Use the Maclaurin series for $\tan^{-1} x$ to find $\tan^{-1} \frac{1}{5}$ and $\tan^{-1} \frac{1}{239}$ to eight-digit accuracy.

   e) Find an approximation to π using the results of parts c) and d) with a note as to the accuracy of your approximation.

2. Find the sum of the series

   $$\sum_{k=2}^{\infty} \ln\left(1 - \frac{1}{k^2}\right)$$

3. This exercise studies series of the form $\sum_{k=1}^{\infty} k^p x^k$.

a) Let $f(x) = \dfrac{1}{1-x}$. Then for $-1 < x < 1$, $f(x) = \sum_{k=0}^{\infty} x^k$. Show that for $-1 < x < 1$

$$xf'(x) = \sum_{k=1}^{\infty} kx^k \tag{1}$$

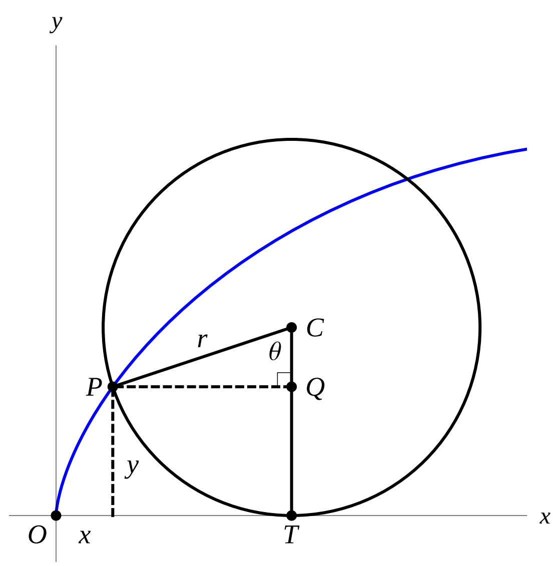
b) Use part a) to find $\sum_{k=1}^{\infty} k \left(\frac{1}{3}\right)^k$.

c) Differentiate both sides of (1) and use that result to find $\sum_{k=1}^{\infty} k^2 x^k$.

d) Find $\sum_{k=1}^{\infty} k^3 x^k$.

e) Find the sum of the series $\sum_{k=1}^{\infty} k^2 \left(\frac{1}{3}\right)^k$ and $\sum_{k=1}^{\infty} k^3 \left(\frac{1}{3}\right)^k$.

4. ***The Cycloid*** The cycloid is the curve generated by a point on a circle that rolls without slipping along a line. Parametric equations can be found for the cycloid by referring to the following figure. We must find the coordinates of P as functions of the parameter θ and the constant r, which is the radius of the rolling circle.



a) Use the fact that the circle has rolled without slipping along the line to conclude that $|OT| = r\theta$ and thus that the coordinates of C are $(r\theta, r)$.

b) Let the coordinates of P be (x, y). Show that $|PQ| = r\sin\theta$, and explain how this implies that $x = r\theta - r\sin\theta = r(\theta - \sin\theta)$.

c) Show that $|QC| = r\cos\theta$, and explain how this implies that $y = r - r\cos\theta = r(1 - \cos\theta)$.