

# STRATEGY FOR TESTING SERIES

MATH 202 and MATH 202 HONORS

We now have several ways of testing a series

$$\sum_{k=1}^{\infty} u_k$$

for convergence or divergence; the problem is to decide which test to use on which series. There are no strict rules to follow in your search for which test will apply to a given series, but here is some advice in the form of a strategy.

We will classify series according to their *form*.

**1.  $p$ -series:** A series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

is called a  $p$ -series. It converges if  $p > 1$ , and diverges if  $p \leq 1$ .

**2. Geometric series:** A series of the form

$$\sum_{k=0}^{\infty} ar^k$$

is called a geometric series. It converges if  $|r| < 1$ , and diverges if  $|r| \geq 1$ .

**3. Alternating Series:** A series of the form

$$\sum_{k=1}^{\infty} (-1)^k a_k \quad \text{or} \quad \sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

is called an alternating series. If an alternating series converges and so does  $\sum a_k$ , we say that the alternating series converges absolutely; if an alternating series converges but  $\sum a_k$  diverges, we say that the alternating series converges conditionally.

These facts being given, we follow this strategy.

**1. Divergence Test:** If you can easily tell (using your intuition and/or L'Hôpital's Rule) that

$$\lim_{k \rightarrow \infty} u_k \neq 0$$

then the series must diverge by the Divergence Test. If the limit is 0, then we can tell nothing about the series by this test.

**2. Alternating Series Test:** If the series you are given is an alternating series, you can use this test. If you can show that  $a_{k+1} \leq a_k$  for all  $k$  large enough and also that  $\lim_{k \rightarrow \infty} a_k = 0$  then the series must converge. To determine whether an alternating series converges conditionally or absolutely, you strip off the  $(-1)^k$  or  $(-1)^{k+1}$  and use one of the other tests on this page. If it looks like the Ratio Test for Absolute Convergence might be effective, you can use it instead of the Alternating Series Test, but the Ratio Test can fail.

**3. Ratio Test (for Absolute Convergence):** If the series involves factorials or other products (including a constant raised to the  $k$ th power), then this test will usually be effective. If the series has both positive and negative terms, let

$$\rho = \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right|$$

If the series has only positive terms, let

$$\rho = \lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k}$$

The series converges (absolutely) if  $\rho < 1$ , and diverges if  $\rho > 1$  or  $\rho = \infty$ . However, if  $\rho = 1$  the test fails and we cannot tell whether the series converges or diverges. Warning: if  $u_k$  has only terms consisting of  $k$  raised to some power, then the Ratio Test will usually fail.

**4. Root Test:** If a positive series involves quantities raised to the  $k$ th or similar powers, then the Root Test will usually be effective. To do the test let

$$\rho = \lim_{k \rightarrow \infty} \sqrt[k]{u_k}.$$

The series converges if  $\rho < 1$ , and diverges if  $\rho > 1$  or  $\rho = \infty$ . However, if  $\rho = 1$  the test fails and we cannot tell whether the series converges or diverges. Warning: if  $u_k$  has only terms consisting of  $k$  raised to some power, then the root test will usually fail.

**5. Limit Comparison Test:** If the positive series  $\sum a_k$  has a form that is similar to the  $p$ -series or to the geometric series, the Limit Comparison Test may be effective. Thus this test will usually work when  $a_k$  has terms consisting of  $k$  raised to some power – exactly when the Ratio and Root Tests fail. To perform this test, you must find a series  $\sum b_k$  that “looks like”  $\sum a_k$ . You must also know whether the series  $\sum b_k$  converges or diverges. Finding this new series is something of an art form, but you can usually succeed by noting what the dominant expressions in the numerator and denominator of  $a_k$  are, then letting those dominant expressions form the numerator and denominator of  $b_k$ . After finding the new series, let

$$\rho = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$$

If you find that  $0 < \rho < \infty$ , then the two series either both converge or both diverge.

**6. Integral Test:** If  $u_k = f(k)$ , where  $f(x)$  is a continuous, positive, and decreasing function for all  $x$  large enough, the integral test applies. You must evaluate

$$\int_1^{\infty} f(x) dx$$

If this improper integral converges, then so does the series  $\sum u_k$ . If the improper integral diverges, then so does the series  $\sum u_k$ .

Here are some examples of how to decide on which test to use. You should complete these examples.

**Example 1.**  $\sum_{k=1}^{\infty} \frac{k-1}{2k+1}$ . Since we can see that  $\lim_{k \rightarrow \infty} u_k = \frac{1}{2}$ , we should use the Divergence Test to conclude that the series diverges.

**Example 2.**  $\sum_{k=1}^{\infty} \frac{\sqrt{k^3+1}}{3k^3+4k^2+2}$ . Since  $a_k$  is a mass of powers of  $k$ , the Limit Comparison Test might be used. The series  $\sum b_k$  to use for comparison should be  $b_k = \frac{\sqrt{k^3}}{k^3} = \frac{1}{k^{3/2}}$ . Here we used the dominant expressions from the numerator and denominator of  $a_k$ .

**Example 3.**  $\sum_{k=1}^{\infty} k e^{-k^2}$ . Since we see powers of  $k$  and a constant raised to the power of  $k$ , we might try the Ratio Test. The Integral Test also works here.

**Example 4.**  $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ . Since this series involves  $k$  as an exponent and  $k!$ , we should try the Ratio Test.

**Example 5.**  $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2+1}$ . Since this is an alternating series, we should try the Alternating Series Test. Note that since  $a_k$  is a bunch of powers of  $k$ , we expect the Ratio Test to fail, and so should avoid it. If we find that the series converges, we must go back and analyze  $\sum_{k=1}^{\infty} \frac{k}{k^2+1}$ . We should use the Limit Comparison Test on this series. If this series converges, the original series converges absolutely; if this series diverges, then the original series converges conditionally.