Chapter 15

Proofs in Linear Algebra

A topic you may very well have studied in geometry, calculus, or physics is vectors. You might recall vectors both in the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ and in 3-space $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. Often one thinks of a vector as a directed line segment from the origin to some other point. Examples of these (both in the plane and in 3-space) are shown in Figure 15.1.

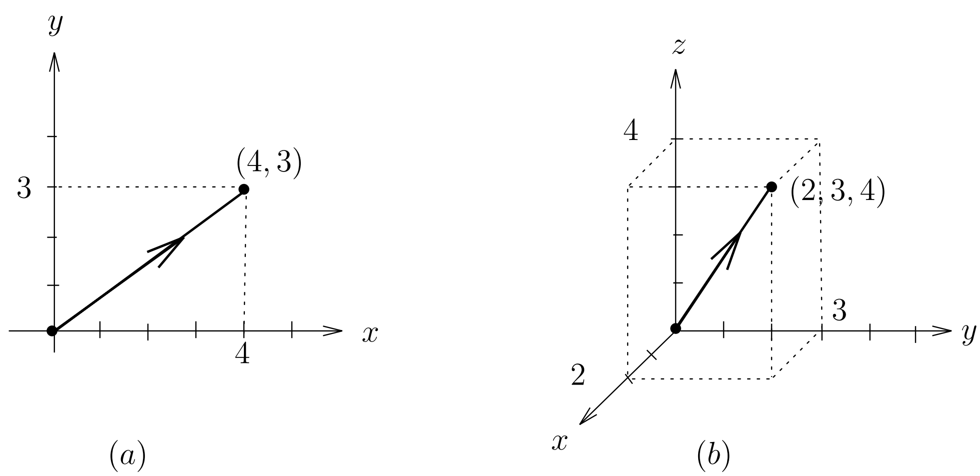

Figure 15.1: Vectors in the plane and 3-space

The vector $\mathbf{u}$ in the plane (it is customary to print vectors in bold) shown in Figure 15.1(a) can be expressed as $\mathbf{u} = (4, 3)$; while the vector $\mathbf{v}$ in 3-space shown in Figure 15.1(b) can be expressed as $\mathbf{v} = (2, 3, 4)$. The vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ in the plane and $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ in 3-space will be of special interest to us.

15.1 Properties of Vectors in 3-Space

One important feature of vectors is that they can be added (to produce another vector); while another is that a vector can be multiplied by an element of some set, usually a real number (again to produce another vector). In this context, these elements are called **scalars**. Let's focus on vectors in 3-space for the present. Let $\mathbf{u} = (a_1, b_1, c_1)$ and $\mathbf{v} = (a_2, b_2, c_2)$, where a_i, b_i, c_i $(i = 1, 2)$ are real numbers. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$$

and the **scalar multiple** of $\mathbf{u}$ by a scalar (real number) α is defined by

$$\alpha\mathbf{u} = (\alpha a_1, \alpha b_1, \alpha c_1).$$

From this definition, it follows that

$$\begin{aligned}\mathbf{u} &= (a_1, b_1, c_1) = (a_1, 0, 0) + (0, b_1, 0) + (0, 0, c_1)\\ &= a_1(1, 0, 0) + b_1(0, 1, 0) + c_1(0, 0, 1) = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}.\end{aligned}$$

That is, it is possible to express a vector $\mathbf{u}$ in 3-space in terms of (and to be called a linear combination of) the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in 3-space. Listed below are eight simple, yet fundamental, properties that follow from these definitions of vector addition and scalar multiplication in $\mathbf{R}^3$:

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$.
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$.
3. For $\mathbf{z} = (0, 0, 0)$, $\mathbf{u} + \mathbf{z} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^3$.
4. For each $\mathbf{u} \in \mathbf{R}^3$, there exists a vector in $\mathbf{R}^3$ which we denote by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{z} = (0, 0, 0)$.
5. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ for all $\alpha \in \mathbf{R}$ and all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$.
6. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ for all $\alpha, \beta \in \mathbf{R}$ and all $\mathbf{u} \in \mathbf{R}^3$.
7. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ for all $\alpha, \beta \in \mathbf{R}$ and all $\mathbf{u} \in \mathbf{R}^3$.
8. $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{R}^3$.

These properties are rather straightforward to verify, as we illustrate with properties 1, 4, and 6. To verify property 1, observe that

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)\\ &= (b_1 + a_1, b_2 + a_2, b_3 + a_3) = \mathbf{v} + \mathbf{u}.\end{aligned}$$

Here, we used only the definition of addition of vectors in $\mathbf{R}^3$ and the fact that addition of real numbers is commutative.

To verify property 4, we begin with a vector $\mathbf{v} = (b_1, b_2, b_3) \in \mathbf{R}^3$ and show that there is some vector in $\mathbf{R}^3$, which we denote by $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{z} = (0, 0, 0)$. There is an obvious choice for $-\mathbf{v}$, however, namely $(-b_1, -b_2, -b_3)$. Observe that

$$\begin{aligned}\mathbf{v} + (-b_1, -b_2, -b_3) &= (b_1, b_2, b_3) + (-b_1, -b_2, -b_3)\\ &= (b_1 + (-b_1),\ b_2 + (-b_2),\ b_3 + (-b_3)) = (0, 0, 0).\end{aligned}$$

Hence, $-\mathbf{v} = (-b_1, -b_2, -b_3)$ has the desired property. We note also that, according to the definition of scalar multiplication in $\mathbf{R}^3$,

$$(-1)\mathbf{v} = ((-1)b_1, (-1)b_2, (-1)b_3) = (-b_1, -b_2, -b_3) = -\mathbf{v}.$$

We will revisit this observation later.

To establish property 6, observe that

$$\begin{aligned}(\alpha+\beta)\mathbf{u} &= (\alpha+\beta)(a_1,b_1,c_1)\\ &= ((\alpha+\beta)a_1,(\alpha+\beta)b_1,(\alpha+\beta)c_1))\\ &= (\alpha a_1+\beta a_1,\alpha b_1+\beta b_1,\alpha c_1+\beta c_1)\\ &= (\alpha a_1,\alpha b_1,\alpha c_1)+(\beta a_1,\beta b_1,\beta c_1)\\ &= \alpha(a_1,b_1,c_1)+\beta(a_1,b_1,c_1)\\ &= \alpha\mathbf{u}+\beta\mathbf{u}.\end{aligned}$$

Thus, showing that $(\alpha+\beta)\mathbf{u} = \alpha\mathbf{u}+\beta\mathbf{u}$ also depends only on some familiar properties of addition and multiplication of real numbers. Vectors in the plane can be added and multiplied by scalars in the expected manner and, in fact, satisfy properties 1-8 as well.

15.2 Vector Spaces

In addition to vectors in the plane and 3-space, there are other mathematical objects that can be added and multiplied by scalars so that properties 1-8 are satisfied. Indeed, these objects provide a generalization of vectors in the plane and 3-space. For this reason, we will refer to these more abstract objects as vectors as well. The study of vectors is a major topic in the area of mathematics called linear algebra.

A nonempty set V, every two elements of which can be added (that is, if $\mathbf{u},\mathbf{v}\in V$, then $\mathbf{u}+\mathbf{v}$ is a unique vector of V) and each element of which can be multiplied by any real number (that is, if $\alpha\in\mathbf{R}$ and $\mathbf{v}\in V$, then $\alpha\mathbf{v}$ is a unique element in V) is called a **vector space** (in fact, a **vector space over R**) if it satisfies the following eight properties:

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$. (Commutative Property)
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$. (Associative Property)
3. There exists an element $\mathbf{z}\in V$ such that $\mathbf{v}+\mathbf{z}=\mathbf{v}$ for all $\mathbf{v}\in V$.
4. For each $\mathbf{v}\in V$, there exists an element $-\mathbf{v}\in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{z}$.
5. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$ for all $\alpha\in\mathbf{R}$ and all $\mathbf{u},\mathbf{v}\in V$.
6. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$ for all $\alpha,\beta\in\mathbf{R}$ and all $\mathbf{v}\in V$.
7. $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$ for $\alpha,\beta\in\mathbf{R}$ and all $\mathbf{v}\in V$.
8. $1\mathbf{v}=\mathbf{v}$ for all $\mathbf{v}\in V$.

The elements of V are called **vectors** and the real numbers in this definition are called **scalars**. Hence if $\mathbf{u},\mathbf{v}\in V$ and $\alpha,\beta\in\mathbf{R}$, then both $\alpha\mathbf{u}$ and $\beta\mathbf{v}$ belong to V. Therefore, $\alpha\mathbf{u}+\beta\mathbf{v}\in V$. The vector $\alpha\mathbf{u}+\beta\mathbf{v}$ is called a **linear combination** of $\mathbf{u}$ and $\mathbf{v}$. We can also discuss linear combinations of more than two vectors. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three vectors in V and let α,β,γ be three scalars (real numbers). Therefore, $\alpha\mathbf{u}$, $\beta\mathbf{v}$, and $\gamma\mathbf{w}$ are three vectors in V and $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. We've now encountered a familiar situation in mathematics. Since addition in V is only defined for two vectors, what exactly is meant by $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$? There are two obvious interpretations of $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$, namely, $(\alpha\mathbf{u}+\beta\mathbf{v})+\gamma\mathbf{w}$ (where $\alpha\mathbf{u}$ and $\beta\mathbf{v}$ are added first, producing the vector $\alpha\mathbf{u}+\beta\mathbf{v}$, which is then added to $\gamma\mathbf{w}$) and $\alpha\mathbf{u}+(\beta\mathbf{v}+\gamma\mathbf{w})$. However, property 2 (the associative law of addition of vectors) guarantees that both interpretations give us the same vector and consequently, there is nothing ambiguous about writing $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}$ without parentheses. In fact, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$, then $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The element $\mathbf{z} \in V$ described in property 3 (and used in property 4) is called a **zero vector** and an element $-\mathbf{v}$ in property 4 is called a **negative** of $\mathbf{v}$. By the commutative property, we also know that $\mathbf{z}+\mathbf{v}=\mathbf{v}$ and $(-\mathbf{v})+\mathbf{v}=\mathbf{z}$ for every vector $\mathbf{v} \in V$. Since V satisfies properties 1–4, the set V forms an abelian group under addition (see Chapter 13).

Although we have only defined a vector space over the set $\mathbf{R}$ of real numbers (and this is all we will deal with), it is not always required that the scalars be real numbers. Indeed, there are certain situations when complex numbers are not only suitable scalars but in fact, the preferred scalars. Other possibilities exist as well.

Of course, we have seen two examples of vector spaces, namely, $\mathbf{R}^2$ and $\mathbf{R}^3$ (with addition and scalar multiplication defined above). More generally, n-space $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}$ (n factors) is a vector space where addition of two vectors $\mathbf{u} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{v} = (b_1, b_2, \ldots, b_n)$ is defined by

$$\mathbf{u}+\mathbf{v} = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$$

and scalar multiplication $\alpha\mathbf{u}$, where $\alpha \in \mathbf{R}$, is defined by

$$\alpha\mathbf{u} = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

We now describe two vector spaces of a very different nature. Recall that $\mathcal{F}_\mathbf{R}$ is the set of all functions from $\mathbf{R}$ to $\mathbf{R}$, that is,

$$\mathcal{F}_\mathbf{R} = \{f \ : \ f : \mathbf{R} \to \mathbf{R}\}.$$

Therefore, the well-known trigonometric function $f_1 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = \sin x$ for all $x \in \mathbf{R}$ belongs to $\mathcal{F}_\mathbf{R}$. The function $f_2 : \mathbf{R} \to \mathbf{R}$ defined by $f_2(x) = 3x + x/(x^2+1)$ for all $x \in \mathbf{R}$ also belongs to $\mathcal{F}_\mathbf{R}$.

For $f, g \in \mathcal{F}_\mathbf{R}$ and a scalar (real number) α, addition and scalar multiplication are defined by

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \quad \text{for all } x \in \mathbf{R}, \\ (\alpha f)(x) &= \alpha(f(x)) \quad \text{for all } x \in \mathbf{R}. \end{aligned}$$

For the functions f_1 and f_2 defined above,

$$(f_1+f_2)(x) = \sin x + 3x + \frac{x}{x^2+1} \quad \text{and} \quad (5f_2)(x) = 15x + \frac{5x}{x^2+1}.$$

Under these definitions of addition and scalar multiplication, $\mathcal{F}_\mathbf{R}$ is a vector space, the verification of which depends only on ordinary addition and multiplication of real numbers. As an illustration, we verify that $\mathcal{F}_\mathbf{R}$ satisfies properties 2-5 of a vector space.

First we verify property 2. Let $f, g, h \in \mathcal{F}_\mathbf{R}$. Then

$$\begin{aligned} ((f+g)+h)(x) &= (f+g)(x)+h(x) = (f(x)+g(x))+h(x) \\ &= f(x)+(g(x)+h(x)) = f(x)+(g+h)(x) \\ &= (f+(g+h))(x) \end{aligned}$$

for all $x \in \mathbf{R}$. Therefore, $(f+g)+h = f+(g+h)$.

Second we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 3 of a vector space. Define the (constant) function $f_0 : \mathbf{R} \to \mathbf{R}$ by $f_0(x) = 0$ for all $x \in \mathbf{R}$. We show that f_0 is a zero vector for $\mathcal{F}_{\mathbf{R}}$. For $f \in \mathcal{F}_{\mathbf{R}}$,

$$(f + f_0)(x) = f(x) + f_0(x) = f(x) + 0 = f(x)$$

for all $x \in \mathbf{R}$. Therefore, $f + f_0 = f$. The function f_0 is called the **zero function** in $\mathcal{F}_{\mathbf{R}}$.

Next we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 4 of a vector space. For each function $f \in \mathcal{F}_{\mathbf{R}}$, define the function $-f : \mathbf{R} \to \mathbf{R}$ by $(-f)(x) = -(f(x))$ for all $x \in \mathbf{R}$. Since

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-f(x)) = 0 = f_0(x)$$

for all $x \in \mathbf{R}$, it follows that $f + (-f) = f_0$ and so $-f$ is a negative of f.

Finally, we show that $\mathcal{F}_{\mathbf{R}}$ satisfies property 5 of a vector space. Let $f, g \in \mathcal{F}_{\mathbf{R}}$ and $\alpha \in \mathbf{R}$. Then, for each $x \in \mathbf{R}$,

$$\begin{aligned}(\alpha(f+g))(x) &= \alpha\left((f+g)(x)\right) = \alpha\left(f(x) + g(x)\right) \\ &= \alpha f(x) + \alpha g(x) = (\alpha f)(x) + (\alpha g)(x) = (\alpha f + \alpha g)(x)\end{aligned}$$

and so $\alpha(f+g) = \alpha f + \alpha g$.

We now consider a special class of real-valued functions defined on $\mathbf{R}$. These functions are important in many areas of mathematics, not only linear algebra. A function $p : \mathbf{R} \to \mathbf{R}$ is called a **polynomial function** (actually a **polynomial function over R**) if

$$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$

for all $x \in \mathbf{R}$, where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers. The expression $p(x)$ itself is called a **polynomial** in x. You may recall that if $a_n \neq 0$, then n is the **degree** of $p(x)$. The zero function f_0 is a polynomial function. It is assigned no degree, however. We denote the set of all polynomial functions over $\mathbf{R}$ by $\mathbf{R}[x]$. Hence $\mathbf{R}[x] \subseteq \mathcal{F}_{\mathbf{R}}$.

Let $f, g \in \mathbf{R}[x]$ and let $\alpha \in \mathbf{R}$. Then

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \quad \text{and} \quad g(x) = b_0 + b_1 x + \ldots + b_m x^m,$$

where n and m are nonnegative integers and $a_i, b_j \in \mathbf{R}$ for $0 \leq i \leq n$ and $0 \leq j \leq m$. If we assume, say, that $m \geq n$, then the sum $f + g$ is the polynomial function defined by

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \ldots + b_m x^m;\end{aligned}$$

while the scalar multiple αf of f by α is the polynomial function defined by

$$(\alpha f)(x) = \alpha(f(x)) = (\alpha a_0) + (\alpha a_1)x + \ldots + (\alpha a_n)x^n.$$

These definitions are, of course, exactly the same as the sum of two elements of $\mathcal{F}_{\mathbf{R}}$ and the scalar product of an element of $\mathcal{F}_{\mathbf{R}}$ by a real number.

Actually, $\mathbf{R}[x]$ is itself a vector space over $\mathbf{R}$ under the addition and scalar multiplication we have just defined. For example, let $f, g \in \mathbf{R}[x]$. Since $\mathbf{R}[x] \subseteq \mathcal{F}_{\mathbf{R}}$ and addition in $\mathbf{R}[x]$ is defined exactly the same as in $\mathcal{F}_{\mathbf{R}}$, it follows that $f + g = g + f$; that is, property 1 of a vector space is satisfied. By the same reasoning, property 2 and properties 5-8 are satisfied as well. The zero function f_0 is in $\mathbf{R}[x]$ and we know that $f + f_0 = f$ for all $f \in \mathcal{F}_{\mathbf{R}}$. Hence $p + f_0 = p$ for all $p \in \mathbf{R}[x]$. So f_0 is a zero vector for $\mathbf{R}[x]$. For $f \in \mathbf{R}[x]$ defined by $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, we know that $-f$ is given by $(-f)(x) = -(f(x)) = (-a_0) + (-a_1)x + \ldots + (-a_n)x^n$. Thus $-f \in \mathbf{R}[x]$ is a negative of f. Thus properties 3 and 4 are satisfied as well, and so $\mathbf{R}[x]$ is a vector space over $\mathbf{R}$.

15.3 Matrices

Among the best known and most important examples of vector spaces are those concerning matrices. A rectangular array of real numbers is called a **matrix**. The plural of "matrix" is "matrices". (In general, a matrix need not be an array of real numbers — it can be a rectangular array of elements from any prescribed set. However, we will deal only with real numbers.) Thus a matrix has m rows and n columns for some pair m, n of positive integers and contains mn real numbers, each of which is located in some row i and column j for integers i and j with $1 \le i \le m$ and $1 \le j \le n$. A matrix with m rows and n columns is said to have **size** $m \times n$ and is called an $m \times n$ **matrix** (read as "m by n matrix"). Hence

$$B = \begin{bmatrix} 1 & \sqrt{2} & -3/2 \\ 0 & -.8 & 4 \end{bmatrix}$$

is a 2×3 matrix, while

$$C = \begin{bmatrix} 4 & 1 & 9 \\ 0 & 3 & 2 \\ 7 & -1 & 1 \end{bmatrix}$$

is a 3×3 matrix. A general $m \times n$ matrix A is commonly written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Therefore, a_{ij} represents the element located in row i and column j of A. This is referred to as the (i, j)**-entry** of A. In fact, it is convenient shorthand notation to represent the matrix A by $[a_{ij}]$ and to write $A = [a_{ij}]$. The ith **row** of A is $[a_{i1}a_{i2} \dots a_{in}]$ and the jth **column** is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}.$$

For two matrices to be equal, they must have the same size. Furthermore, two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal**, written as $A = B$, if $a_{ij} = b_{ij}$ for all integers i and j with $1 \le i \le m$ and $1 \le j \le n$. That is, $A = B$ if A and B have the same size and corresponding entries are equal. Hence, in order for

$$A = \begin{bmatrix} 2 & x & -3 \\ 1/2 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4/5 & -3 \\ y & 4 & 0 \end{bmatrix}$$

to be equal, we must have $x = 4/5$ and $y = 1/2$.

For positive integers m and n, let $M_{mn}[\mathbf{R}]$ denote the set of all $m \times n$ matrices whose entries are real numbers. If $m = n$, then the matrices are called **square matrices**. The set of all $m \times m$ (square) matrices whose entries are real numbers is also denoted by $M_m[\mathbf{R}]$.

We now define addition and scalar multiplication in $M_{mn}[\mathbf{R}]$. Let $A, B \in M_{mn}[\mathbf{R}]$, where $A = [a_{ij}]$ and $B = [b_{ij}]$. The **sum** $A + B$ of A and B is defined as that $m \times n$ matrix $[c_{ij}]$, where $c_{ij} = a_{ij} + b_{ij}$ for all integers i and j with $1 \le i \le m$ and $1 \le j \le n$. For $\alpha \in \mathbf{R}$, the **scalar multiple** αA of A by α is defined as $\alpha A = [d_{ij}]$, where $d_{ij} = \alpha a_{ij}$ for all integers i and j with $1 \le i \le m$ and $1 \le j \le n$. For example, if

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -9 & 2 \\ -2 & 5 & 0 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 5 & -10 & -1 \\ -2 & 9 & 0 \end{bmatrix} \quad \text{and} \quad (-2)A = \begin{bmatrix} -4 & 2 & 6 \\ 0 & -8 & 0 \end{bmatrix}.$$

Under this addition and scalar multiplication, $M_{mn}[\mathbf{R}]$ is a vector space. As an illustration, we verify that properties 1 and 3-5 of a vector space are satisfied in $M_2[\mathbf{R}]$. Let $\alpha \in \mathbf{R}$ and let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then

$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = B + A. \end{aligned}$$

This verifies property 1 of a vector space. We see here that verifying property 1 depended only on the definition of addition of matrices and the fact that real numbers are commutative under addition.

Let $Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, often called the 2×2 **zero matrix**. Then

$$\begin{aligned} A + Z &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} + 0 & a_{12} + 0 \\ a_{21} + 0 & a_{22} + 0 \end{bmatrix} \\ &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A \end{aligned}$$

and so Z is a zero element of $M_2[\mathbf{R}]$, thereby verifying property 3.

Next, let $-A = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}$. Consequently,

$$A + (-A) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = Z,$$

and so $-A$ is a negative of A. Therefore, property 4 is satisfied. We note also that if A is multiplied by the scalar -1, then we obtain

$$(-1)A = (-1) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} = -A.$$

Finally,

$$\alpha(A + B) = \alpha \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} \alpha(a_{11} + b_{11}) & \alpha(a_{12} + b_{12}) \\ \alpha(a_{21} + b_{21}) & \alpha(a_{22} + b_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} + \alpha b_{11} & \alpha a_{12} + \alpha b_{12} \\ \alpha a_{21} + \alpha b_{21} & \alpha a_{22} + \alpha b_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \alpha b_{12} \\ \alpha b_{21} & \alpha b_{22} \end{bmatrix}$$

$$= \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \alpha \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \alpha A + \alpha B.$$

Under the right set of circumstances, matrices can also be multiplied — although this is, of course, not a requirement for a vector space.

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times r$ matrix, that is, let A and B be two matrices, where the number of columns in A equals the number of rows in B. In this case, we define the **product** AB of A and B as that $m \times r$ matrix $[c_{ij}]$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj} \qquad (15.1)$$

for all integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq r$. Hence the (i, j)-entry of AB is obtained from the ith row of A and jth column of B, that is,

$$[a_{i1}\ a_{i2}\ \ldots\ a_{in}] \quad \text{and} \quad \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

by multiplying corresponding terms of this row and column and then adding all n products. The expression (15.1) is referred to as the **inner product** of the ith row of A and the jth column of B. For example, let

$$A = \begin{bmatrix} 1 & -3 & 5 & 0 \\ -1 & 0 & 6 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -6 & 5 \\ 2 & 0 & 1 \\ 3 & 3 & 2 \\ -6 & 9 & 0 \end{bmatrix}.$$

Since A is a 2×4 matrix and B is a 4×3 matrix, the product AB is defined and, in fact, $AB = [c_{ij}]$ is the 2×3 matrix, where the six inner products are

$$\begin{aligned} c_{11} &= 1 \cdot 1 + (-3) \cdot 2 + 5 \cdot 3 + 0 \cdot (-6) = 10 \\ c_{12} &= 1 \cdot (-6) + (-3) \cdot 0 + 5 \cdot 3 + 0 \cdot 9 = 9 \\ c_{13} &= 1 \cdot 5 + (-3) \cdot 1 + 5 \cdot 2 + 0 \cdot 0 = 12 \\ c_{21} &= (-1) \cdot 1 + 0 \cdot 2 + 6 \cdot 3 + 2 \cdot (-6) = 5 \\ c_{22} &= (-1) \cdot (-6) + 0 \cdot 0 + 6 \cdot 3 + 2 \cdot 9 = 42 \\ c_{23} &= (-1) \cdot 5 + 0 \cdot 1 + 6 \cdot 2 + 2 \cdot 0 = 7. \end{aligned}$$

Hence

$$AB = \begin{bmatrix} 10 & 9 & 12 \\ 5 & 42 & 7 \end{bmatrix}.$$

On the other hand, since the matrix B above is a 4×3 matrix and A is a 2×4 matrix, the product BA is not defined. Certainly, however, if A and B are any two square matrices of the same size, then AB and BA are both defined though they need not be equal. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \text{ while } \quad BA = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

15.4 Some Properties of Vector Spaces

Although we have now seen several different vector spaces, there are a number of properties that these vector spaces have in common (in addition to the eight defining properties). Indeed, there are a number of additional properties that *all* vector spaces have in common. Since vector spaces are defined by eight properties, one might expect, and rightfully so, that any other properties they have in common are consequences of these eight properties.

According to property 3, every vector space contains at least one zero vector and by property 4, every vector has at least one negative. We show that "at least one" can be replaced by "exactly one" in both instances. Actually, these are consequences of the fact that every vector space is a group under addition (Chapter 13). We verify these nevertheless.

Theorem 15.1 *Every vector space has a unique zero vector.*

Proof. Let V be a vector space and assume that $\mathbf{z}$ and $\mathbf{z}'$ are both zero vectors in V. Since $\mathbf{z}$ is a zero vector, $\mathbf{z}' + \mathbf{z} = \mathbf{z}'$. Moreover, since $\mathbf{z}'$ is a zero vector, $\mathbf{z} + \mathbf{z}' = \mathbf{z}$. Therefore, $\mathbf{z} = \mathbf{z} + \mathbf{z}' = \mathbf{z}' + \mathbf{z} = \mathbf{z}'$. ∎

As a consequence of Theorem 15.1, we now know that a vector space V possesses only one zero vector $\mathbf{z}$ that satisfies property 3 of a vector space. Hence we can now refer to $\mathbf{z}$ as *the* zero vector of V.

Theorem 15.2 *Let V be a vector space. Then every vector in V has a unique negative.*

Proof. Let $\mathbf{v} \in V$ and assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both negatives of $\mathbf{v}$. Thus $\mathbf{v} + \mathbf{v}_1 = \mathbf{z}$ and $\mathbf{v} + \mathbf{v}_2 = \mathbf{z}$. Hence

$$\mathbf{v}_1 = \mathbf{v}_1 + \mathbf{z} = \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 = \mathbf{z} + \mathbf{v}_2 = \mathbf{v}_2. \qquad ∎$$

Proof Analysis Let's revisit the proof of Theorem 15.2. We wanted to show that each vector $\mathbf{v}$ has only one negative. We assumed that there were two negatives of $\mathbf{v}$, namely $\mathbf{v}_1$ and $\mathbf{v}_2$. Our goal then was to show that $\mathbf{v}_1 = \mathbf{v}_2$. We started with $\mathbf{v}_1$. Our idea was to add $\mathbf{z}$ to $\mathbf{v}_1$, as this sum is the vector $\mathbf{v}_1$ again. Since $\mathbf{z}$ can also be expressed as $\mathbf{v} + \mathbf{v}_2$, we made this substitution, bringing the vector $\mathbf{v}_2$ into the discussion. Eventually, we showed that this expression for $\mathbf{v}_1$ was also equal to $\mathbf{v}_2$. There is another approach we could have tried.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are both negatives of $\mathbf{v}$, it follows that $\mathbf{v} + \mathbf{v}_1 = \mathbf{z}$ and $\mathbf{v} + \mathbf{v}_2 = \mathbf{z}$, that is, $\mathbf{v} + \mathbf{v}_1 = \mathbf{v} + \mathbf{v}_2$. If we add the same vector to both $\mathbf{v} + \mathbf{v}_1$ and $\mathbf{v} + \mathbf{v}_2$, we obtain equal vectors (since $\mathbf{v} + \mathbf{v}_1 = \mathbf{v} + \mathbf{v}_2$). A good choice of a vector to add to both $\mathbf{v} + \mathbf{v}_1$ and $\mathbf{v} + \mathbf{v}_2$ is a negative of $\mathbf{v}$ (either one!). This gives us the following list of equalities:

$$\begin{aligned} \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_1) &= \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) \\ (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_1 &= (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 \\ \mathbf{z} + \mathbf{v}_1 &= \mathbf{z} + \mathbf{v}_2 \\ \mathbf{v}_1 &= \mathbf{v}_2. \end{aligned}$$

Although this string of equalities results in $\mathbf{v}_1 = \mathbf{v}_2$, this is not a particularly well-written proof. However, since our goal is to show that $\mathbf{v}_1 = \mathbf{v}_2$, this suggests a way to arrive at our goal. We start with $\mathbf{v}_1$ (at the bottom of the left column), proceed upward, then to the right, and downward, producing

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{z} + \mathbf{v}_1 = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_1 = \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_1) \\ &= \mathbf{v}_1 + (\mathbf{v} + \mathbf{v}_2) = (\mathbf{v}_1 + \mathbf{v}) + \mathbf{v}_2 = \mathbf{z} + \mathbf{v}_2 = \mathbf{v}_2, \end{aligned}$$

which is similar to the proof given in Theorem 15.2 (though a bit longer). $\diamondsuit$

As a consequence of Theorem 15.2, we can now refer to $-\mathbf{v}$ as *the* negative of $\mathbf{v}$. Of course, the zero vector $\mathbf{z}$ has the property that $\mathbf{z}+\mathbf{z} = \mathbf{z}$. However, no other vector has this property.

Theorem 15.3 *Let V be a vector space. If $\mathbf{v}$ is a vector such that $\mathbf{v}+\mathbf{v} = \mathbf{v}$, then $\mathbf{v} = \mathbf{z}$.*

Proof. Since $\mathbf{v} + (-\mathbf{v}) = \mathbf{z}$, it follows that

$$\mathbf{z} = \mathbf{v} + (-\mathbf{v}) = (\mathbf{v} + \mathbf{v}) + (-\mathbf{v}) = \mathbf{v} + (\mathbf{v} + (-\mathbf{v})) = \mathbf{v} + \mathbf{z} = \mathbf{v}. \qquad \blacksquare$$

A proof like that given for Theorem 15.3 can be obtained by adding $-\mathbf{v}$ to the equal vectors $\mathbf{v} + \mathbf{v}$ and $\mathbf{v}$ and proceeding as we did in the discussion following the proof of Theorem 15.2. Also, see Exercise 15.6(b).

We now describe two other properties concerning the zero vector that are consequences of Theorem 15.3.

Corollary 15.4 *Let V be a vector space. Then*

(*i*) $0\mathbf{v} = \mathbf{z}$ *for every vector* $\mathbf{v}$ *in* V *and*

(*ii*) $\alpha\mathbf{z} = \mathbf{z}$ *for every scalar* $\alpha \in \mathbf{R}$.

Proof. First, we prove (*i*). Observe that

$$0\mathbf{v} = (0 + 0)\mathbf{v} = 0\mathbf{v} + 0\mathbf{v}.$$

By Theorem 15.3, $0\mathbf{v} = \mathbf{z}$.

Next we verify (*ii*). Observe that

$$\alpha\mathbf{z} = \alpha(\mathbf{z} + \mathbf{z}) = \alpha\mathbf{z} + \alpha\mathbf{z}.$$

Again, by Theorem 15.3, $\alpha\mathbf{z} = \mathbf{z}$. $\blacksquare$

Hence, by Corollary 15.4, $0\mathbf{v} = \mathbf{z}$ for every vector $\mathbf{v}$ in a vector space and $\alpha\mathbf{z} = \mathbf{z}$ for every scalar α. That is, if either $\alpha = 0$ or $\mathbf{v} = \mathbf{z}$, then $\alpha\mathbf{v} = \mathbf{z}$. We now show that the converse of this statement is true as well.

Theorem 15.5 *Let V be a vector space. If $\alpha\mathbf{v} = \mathbf{z}$, then either $\alpha = 0$ or $\mathbf{v} = \mathbf{z}$.*

Proof. If $\alpha = 0$, then, of course, the statement is true. So we may assume that $\alpha \neq 0$. In this case,

$$\mathbf{v} = 1\mathbf{v} = \left(\frac{1}{\alpha}\,\alpha\right)\mathbf{v} = \left(\frac{1}{\alpha}\right)(\alpha\mathbf{v}) = \left(\frac{1}{\alpha}\right)\mathbf{z} = \mathbf{z}. \qquad \blacksquare$$

Another useful property is that the scalar multiple of a vector by -1 is the negative of that vector. Actually, we have observed this earlier with two particular vector spaces but this is true in general.

Theorem to Prove If $\mathbf{v}$ is a vector in a vector space, then $(-1)\mathbf{v} = -\mathbf{v}$.

Proof Strategy Since $\mathbf{v}$ has a unique negative, to show that $(-1)\mathbf{v} = -\mathbf{v}$, we need only verify that the sum of $\mathbf{v}$ and $(-1)\mathbf{v}$ is $\mathbf{z}$. $\diamondsuit$

Theorem 15.6 *If $\mathbf{v}$ is a vector in a vector space, then $(-1)\mathbf{v} = -\mathbf{v}$.*

Proof. Observe that

$$\mathbf{v} + (-1)\mathbf{v} = 1\mathbf{v} + (-1)\mathbf{v} = (1 + (-1))\mathbf{v} = 0\mathbf{v} = \mathbf{z}.$$

Hence $(-1)\mathbf{v} = -\mathbf{v}$. $\blacksquare$

15.5 Subspaces

Earlier we saw that $\mathcal{F}_{\mathbf{R}} = \{f \ : \ f : \mathbf{R} \to \mathbf{R}\}$ is a vector space (under function addition and scalar multiplication). Since the set $\mathbf{R}[x]$ of all polynomial functions over $\mathbf{R}$ is a subset of $\mathcal{F}_{\mathbf{R}}$ and the addition and scalar multiplication defined in $\mathbf{R}[x]$ are exactly the same as those defined in $\mathcal{F}_{\mathbf{R}}$, it was considerably easier to show that $\mathbf{R}[x]$ is a vector space. This idea can be made more general.

For a vector space V, a subset W of V is called a **subspace** of V if W is vector space under the same addition and scalar multiplication defined on V. Hence if W is a subspace of a known vector space V, then W itself is a vector space. Since every subspace contains a zero vector, W must be nonempty.

As we study vector spaces further, we will see that certain subspaces appear regularly and consequently it is beneficial to have an understanding of subspaces. Furthermore, some sets having an addition and scalar multiplication defined on them are subsets of known vector spaces and can be shown to be vector spaces more easily by verifying that they are subspaces.

What is required to show that a subset W of a vector space V is a subspace of V? Of course, W must satisfy the eight properties required of all vector spaces. In addition, if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u} + \mathbf{v}$ must belong to W. This property is expressed by saying that W is **closed under addition**. Also, if α is a scalar (a real number) and $\mathbf{v} \in W$, then $\alpha\mathbf{v}$ must belong to W. We express this property by saying that W is **closed under scalar multiplication**.

Property 1 (the commutative property) requires that $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ for every two vectors $\mathbf{u}$ and $\mathbf{v}$ in W. However, V is a vector space and satisfies property 1. Thus $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ and W satisfies property 1. By the same reasoning, property 2 and properties 5-8 are satisfied by W. These properties of W are said to be **inherited** from V. Hence for a nonempty subset W of a vector space V to be a subspace of V, it is necessary that W be closed under addition and scalar multiplication. Perhaps surprisingly, these requirements are sufficient as well for a nonempty subset W of V to be subspace of V.

Theorem 15.7 **(The Subspace Test)** *A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under addition and scalar multiplication.*

Proof. First, let W be a subspace of V. Certainly, W is closed under addition and scalar multiplication. For the converse, let W be a nonempty subset of V that is closed under addition and scalar multiplication. As we noted earlier, W inherits properties 1, 2 and 5-8 of a vector space from V. Since W is nonempty and is closed under addition and scalar multiplication, only properties 3 and 4 remain to be verified. Since $W \neq \emptyset$, there is some vector $\mathbf{v}$ in W. Since W is closed under scalar multiplication, it follows by Corollary 15.4(i) that $0\mathbf{v} = \mathbf{z} \in W$. Hence W contains a zero vector (namely the zero vector of V) and property 3 is satisfied. Now let $\mathbf{w}$ be any vector of W. Again, $(-1)\mathbf{w} \in W$. However, by Theorem 15.6, $(-1)\mathbf{w} = -\mathbf{w} \in W$, and so $\mathbf{w}$ has a negative in W (namely the negative of $\mathbf{w}$ in V). Thus property 4 is satisfied in W as well. ■

The proof of Theorem 15.7 brought out two important facts. Namely, if W is a subspace of a vector space V, then W contains a zero vector (namely, the zero vector of V) and for every vector $\mathbf{w} \in W$, its negative $-\mathbf{w}$ belongs to W as well.

Every vector space V (containing at least two elements) always contains two subspaces, namely V itself and the subspace consisting only of the zero vector of V. We now present several examples to illustrate how the Subspace Test (Theorem 15.7) can be applied to show that certain subsets of a vector space are (or are not) subspaces of that vector space. The first two examples concern the vector space $\mathbf{R}^3$.

Result 15.8 *The set*

$$W = \{(a, b, 2a - b) \ : \ a, b \in \mathbf{R}\}$$

is a subspace of $\mathbf{R}^3$.

First observe that W contains all vectors of $\mathbf{R}^3$ whose 3rd coordinate is twice the first coordinate minus the second coordinate. So for example, W contains $(3, 2, 4)$, taking $a = 3$ and $b = 2$, and $(0, 0, 0)$, taking $a = b = 0$. Of course, if W is to be a subspace of $\mathbf{R}^3$, then it is essential that W contains the zero vector of $\mathbf{R}^3$.

Proof of Result 15.8. Since W contains the zero vector of $\mathbf{R}^3$, it follows that $W \neq \emptyset$. To show that W is a subspace of V, we need only show that W is closed under addition (that is, if $\mathbf{u}, \mathbf{v} \in W$, then $\mathbf{u}+\mathbf{v} \in W$) and that W is closed under scalar multiplication (that is, if $\mathbf{u} \in W$ and $\alpha \in \mathbf{R}$, then $\alpha\mathbf{u} \in W$). Let $\mathbf{u}, \mathbf{v} \in W$ and $\alpha \in \mathbf{R}$. Then $\mathbf{u} = (a, b, 2a - b)$ and $\mathbf{v} = (c, d, 2c - d)$, where $a, b, c, d \in \mathbf{R}$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (a + c, b + d, 2(a + c) - (b + d)) \in W \quad \text{and} \\ \alpha\mathbf{u} &= (\alpha a, \alpha b, 2(\alpha a) - (\alpha b)) \in W. \end{aligned}$$

By the Subspace Test, W is a subspace of $\mathbf{R}^3$. ■

Example 15.9 *Determine whether*

$$W = \{(a, b, a^2 + b) \ : \ a, b \in \mathbf{R}\}$$

is a subspace of $\mathbf{R}^3$.

Solution. Taking $a = b = 1$, we see that $\mathbf{u} = (1, 1, 2) \in W$. Then $2\mathbf{u} = (2, 2, 4)$. Since $4 \neq 2^2 + 2$, it follows that $2\mathbf{u} \notin W$. Since W is not closed under scalar multiplication, W is not a subspace of $\mathbf{R}^3$. (The subset W of $\mathbf{R}$ is not closed under addition either since $\mathbf{u} + \mathbf{u} \notin W$.) $\diamondsuit$

We next consider the vector space $\mathcal{F}_{\mathbf{R}}$. We have already mentioned that $\mathbf{R}[x]$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. Also, the set $\mathcal{C}_{\mathbf{R}} = \{f \in \mathcal{F}_{\mathbf{R}} : f \text{ is continuous}\}$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. Indeed, $\mathbf{R}[x]$ is a subspace of $\mathcal{C}_{\mathbf{R}}$ as well.

Result 15.10 *Let* $\mathcal{F}_0 = \{f \in \mathcal{F}_{\mathbf{R}} : f(1) = 0\}$. *Then* $\mathcal{F}_0$ *is a subspace of* $\mathcal{F}_{\mathbf{R}}$.

Hence the function $f_1 : \mathbf{R} \to \mathbf{R}$ defined by $f_1(x) = x - 1$ belongs to $\mathcal{F}_0$, as does the zero function $f_0 : \mathbf{R} \to \mathbf{R}$ defined by $f_0(x) = 0$ for all x.

Proof of Result 15.10. Since $\mathcal{F}_0$ contains the zero function, $\mathcal{F}_0 \neq \emptyset$. Let $f, g \in \mathcal{F}_0$ and $\alpha \in \mathbf{R}$. Then

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0 \quad \text{and} \quad (\alpha f)(1) = \alpha f(1) = \alpha \cdot 0 = 0.$$

Thus $f + g \in \mathcal{F}_0$ and $\alpha f \in \mathcal{F}_0$. By the Subspace Test, $\mathcal{F}_0$ is a subspace of $\mathcal{F}_{\mathbf{R}}$. ■

Example 15.11 *Determine whether*

$$\mathcal{F}_1 = \{f \in \mathcal{F}_{\mathbf{R}} : f(0) = 1\}$$

is a subspace of $\mathcal{F}_{\mathbf{R}}$.

Solution. Observe that the functions $g, h \in \mathcal{F}_{\mathbf{R}}$ defined by $g(x) = x+1$ and $h(x) = x^2+1$ belong to $\mathcal{F}_1$. However, $(g+h)(x) = g(x)+h(x) = x^2+x+2$ and $(g+h)(0) = 2$, so $g+h \notin \mathcal{F}_1$. Therefore, $\mathcal{F}_1$ is not a subspace of $\mathcal{F}_{\mathbf{R}}$. $\diamondsuit$

The next example concerns the vector space $M_2(\mathbf{R})$ of 2×2 matrices with real entries.

Result 15.12 *The set*

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : a, b, c \in \mathbf{R} \right\}$$

is a subspace of $M_2(\mathbf{R})$.

Hence W consists of all these 2×2 matrices whose $(1, 2)$-entry is 0. Thus the zero matrix, all of whose entries are 0, belongs to W.

Proof of Result 15.12. Since W contains the zero matrix, $W \neq \emptyset$. Let $A, B \in W$ and $\alpha \in \mathbf{R}$. So

$$A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & 0 \\ e & f \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbf{R}$. Then

$$A + B = \begin{bmatrix} a+d & 0 \\ b+e & c+f \end{bmatrix} \quad \text{and} \quad \alpha A = \begin{bmatrix} \alpha a & 0 \\ \alpha b & \alpha c \end{bmatrix}.$$

Therefore, $A + B$ and αA belong to W and by the Subspace Test, W is a subspace of $M_2(\mathbf{R})$. ■

15.6 Spans of Vectors

In Result 15.12 we showed that the set

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \ a, b, c \in \mathbf{R} \right\}$$

is a subspace of $M_2(\mathbf{R})$. Thus if $A \in W$, then $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ for some $a, b, c \in \mathbf{R}$. Observe, also, that

$$\begin{aligned} A &= \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \\ &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

In other words, A (and, consequently, every matrix in W) is a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, W is the set of all linear combinations of these three matrices. This observation illustrates a more general situation.

Recall that if V is a vector space, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$, then every vector of the form $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Thus, by taking $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, we see that the zero vector is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Also, by taking $\alpha_i = 1$ for a fixed integer i $(1 \le i \le n)$ and all other scalars 0, we see that each vector $\mathbf{v_i}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We have noted that every linear combination of vectors in V is a vector in V and, of course, the set of all such linear combinations is a subset of V. In fact, more can be said of this subset.

Theorem 15.13 *Let V be a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. Then the set W of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a subspace of V.*

Proof. Since W contains the zero vector of V, it follows that $W \neq \emptyset$. Let $\mathbf{u}, \mathbf{w} \in W$ and let $\alpha \in \mathbf{R}$. Then $\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n$ and $\mathbf{w} = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \ldots + \beta_n\mathbf{v}_n$, where $\alpha_i, \beta_i \in \mathbf{R}$ for $1 \le i \le n$. Then

$$\begin{aligned} \mathbf{u} + \mathbf{w} &= (\alpha_1 + \beta_1)\mathbf{v}_1 + (\alpha_2 + \beta_2)\mathbf{v}_2 + \ldots + (\alpha_n + \beta_n)\mathbf{v}_n \text{ and} \\ \alpha\mathbf{u} &= (\alpha\alpha_1)\mathbf{v}_1 + (\alpha\alpha_2)\mathbf{v}_2 + \ldots + (\alpha\alpha_n)\mathbf{v}_n. \end{aligned}$$

So both $\mathbf{u} + \mathbf{w}$ and $\alpha\mathbf{u}$ are linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and hence belong to W. Thus by the Subspace Test, W is a subspace of V. ■

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in a vector space V, the subspace W of V consisting of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and is denoted by $\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. Also, W is referred to as the subspace of V **spanned** by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

By Result 15.12,

$$W = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} : \ a, b, c \in \mathbf{R} \right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle.$$

We saw in Result 15.8 that $W = \{(a, b, 2a - b) \ : \ a, b \in \mathbf{R}\}$ is a subspace of of $\mathbf{R}^3$. Since $(a, b, 2a - b) = a(1, 0, 2) + b(0, 1, -1)$, it follows that W is spanned by the vectors $(1, 0, 2)$ and $(0, 1, -1)$, that is, $W = \langle (1, 0, 2), (0, 1, -1) \rangle$.

We consider another illustration of spans of vectors.

Result 15.14 *Let f_1, f_2, f_3, g_2 and g_3 be five functions in $\mathbf{R}[x]$ defined by $f_1(x) = 1$, $f_2(x) = 1 + x^2$, $f_3(x) = 1 + x^2 + x^4$, $g_2(x) = x^2$, and $g_3(x) = x^4$ for all $x \in \mathbf{R}$, and let $W = \langle f_1, f_2, f_3 \rangle$ and $W' = \langle f_1, g_2, g_3 \rangle$. Then $W = W'$.*

Since W and W' are sets of vectors (polynomial functions) and our goal is to show that $W = W'$, we proceed in the standard manner by showing that each of W and W' is a subset of the other.

Proof of Result 15.14. First, we show that $W \subseteq W'$. Let $f \in W$. Then $f = af_1 + bf_2 + cf_3$ for some $a, b, c \in \mathbf{R}$. Hence, for each $x \in \mathbf{R}$,

$$\begin{aligned} f(x) &= a \cdot 1 + b \cdot \left(1 + x^2\right) + c \cdot \left(1 + x^2 + x^4\right) \\ &= (a + b + c) + (b + c) \cdot x^2 + c \cdot x^4. \end{aligned}$$

Thus, f is also a linear combination of f_1, g_2, and g_3. Consequently, $W \subseteq W'$. It remains to show that $W' \subseteq W$. Let $g \in W'$. Then

$$g = af_1 + bg_2 + cg_3 \text{ for some } a, b, c \in \mathbf{R}.$$

So, for each $x \in \mathbf{R}$,

$$\begin{aligned} g(x) &= a \cdot 1 + b \cdot x^2 + c \cdot x^4 = (a - b) \cdot 1 + b \cdot \left(1 + x^2\right) + c \cdot x^4 \\ &= (a - b) \cdot 1 + (b - c) \cdot \left(1 + x^2\right) + c \cdot \left(1 + x^2 + x^4\right). \end{aligned}$$

Hence g is also a linear combination of f_1, f_2, f_3 as well and so $W' \subseteq W$. ■

From what we have seen, if V is a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, then $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ is a subspace of V (that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$). Quite possibly other subspaces of V contain $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ as well. Of course, V itself is a subspace of V containing $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. In a certain sense though, W is the smallest subspace of V containing $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Theorem 15.15 *Let V be a vector space containing the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ and let $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$. If W' is a subspace of V containing $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$, then W is a subspace of W'.*

Proof. Since W and W' are subspaces of V, we need only show that $W \subseteq W'$. Let $\mathbf{v} \in W$. Thus $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$, where $\alpha_i \in \mathbf{R}$ for $1 \le i \le n$. Since $\mathbf{v}_i \in W'$ for $1 \le i \le n$ and W' is a subspace of V, it follows that $\mathbf{v} \in W'$. Hence $W \subseteq W'$. ■

There is a consequence of Theorem 15.15 that is especially useful.

Corollary 15.16 *Let V be a vector space spanned by the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$. If W is a subspace of V containing $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$, then $W = V$.*

Proof. Since W is a subspace of V, certainly $W \subseteq V$. By Theorem 15.15, $V \subseteq W$. Thus $W = V$. ■

To illustrate a number of the concepts and results introduced thus far, we consider an example concerning 3-space.

Result 15.17

(*i*) *For the vectors* $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, *and* $\mathbf{k} = (0,0,1)$, $\mathbf{R}^3 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$.

(*ii*) *If* $\mathbf{w}_1 = (1,1,0)$, $\mathbf{w}_2 = (0,1,1)$, *and* $\mathbf{w}_3 = (1,1,1)$, *then* $\mathbf{R}^3 = \langle \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \rangle$.

(*iii*) *Let* $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (1,1,0)$, *and* $\mathbf{u}_3 = (0,0,1)$. *Then* $\langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$.

Proof. Let $W_1 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$. Since W_1 is a subspace of $\mathbf{R}^3$, it follows that $W_1 \subseteq \mathbf{R}^3$. We now show that $\mathbf{R}^3 \subseteq W_1$. Let $\mathbf{v} \in \mathbf{R}^3$. So $\mathbf{v} = (a,b,c)$, where $a,b,c \in \mathbf{R}$. Then $\mathbf{v} = (a,0,0) + (0,b,0) + (0,0,c) = a(1,0,0) + b(0,1,0) + c(0,0,1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Hence $\mathbf{v}$ is a linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, and so $\mathbf{v} \in W_1$. Hence $\mathbf{R}^3 \subseteq W_1$. This implies that $\mathbf{R}^3 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ and (*i*) is verified.

Next, we verify (*ii*). Let $W_2 = \langle \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \rangle$. To verify that $\mathbf{R}^3 = W_2$, it suffices to show by Corollary 15.16 and part (*i*) of this result that each of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ belongs to W_2. To show that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ belong to W_2, we are then required to show that each of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is a linear combination of $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$. Since $\mathbf{i} = (1,0,0) = (1,1,1) + (-1)(0,1,1)$, it follows that $\mathbf{i} = 0 \cdot \mathbf{w}_1 + (-1)\mathbf{w}_2 + 1 \cdot \mathbf{w}_3$. Now $\mathbf{j} = (0,1,0) = (1,1,0) + (0,1,1) + (-1)(1,1,1)$; so $\mathbf{j} = 1 \cdot \mathbf{w}_1 + 1 \cdot \mathbf{w}_2 + (-1)\mathbf{w}_3$. Finally, $\mathbf{k} = (0,0,1) = (1,1,1) + (-1)(1,1,0)$ and so $\mathbf{k} = (-1)\mathbf{w}_1 + 0 \cdot \mathbf{w}_2 + 1 \cdot \mathbf{w}_3$. Hence $\mathbf{R}^3 = W_2$ and (ii) is established.

Finally, we verify (*iii*). Let $W = \langle \mathbf{u}_1, \mathbf{u}_2 \rangle$ and $W' = \langle \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \rangle$. Since W' contains the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, it follows by Theorem 15.15 that $W \subseteq W'$.

By Corollary 15.16, to prove that $W' \subseteq W$, we need only show that each of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ belongs to W, that is, each of these three vectors is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. This is obvious for $\mathbf{u}_1$ and $\mathbf{u}_2$ as $\mathbf{u}_1 = 1 \cdot \mathbf{u}_1 + 0 \cdot \mathbf{u}_2$ and $\mathbf{u}_2 = 0 \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2$. Thus it remains only to show that $\mathbf{u}_3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. However, $\mathbf{u}_3 = (0,0,1) = (1,1,1) + (-1)(1,1,0) = 1 \cdot \mathbf{u}_1 + (-1)\mathbf{u}_2$, completing the proof. ■

15.7 Linear Dependence and Independence

For the vectors $\mathbf{u}_1 = (1,1,0)$ and $\mathbf{u}_2 = (0,1,1)$ in $\mathbf{R}^3$, the vector $\mathbf{u}_3 = (-1,1,2) \in \mathbf{R}^3$ is a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ since

$$\mathbf{u}_3 = (-1,1,2) = (-1) \cdot \mathbf{u}_1 + 2 \cdot \mathbf{u}_2 = (-1) \cdot (1,1,0) + 2 \cdot (0,1,1).$$

Therefore, in a certain sense, the vector $\mathbf{u}_3$ *depends on* $\mathbf{u}_1$ and $\mathbf{u}_2$ *in a linear manner.* This linear dependence can be restated as

$$(-1) \cdot \mathbf{u}_1 + 2 \cdot \mathbf{u}_2 + (-1) \cdot \mathbf{u}_3 = (0,0,0).$$

This kind of dependence plays an important role in linear algebra.

Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be a nonempty set of vectors in a vector space V. The set S is called **linearly dependent** if there exist scalars $c_1, c_2, \ldots, c_m$, not all 0, such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{z}$. If S is not linearly dependent, then S is said to be **linearly independent**. For $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, we also say that the *vectors* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are linearly dependent or linearly independent according to whether the *set* S is linearly dependent or linearly independent, respectively. Consequently, the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$ are **linearly independent** if whenever $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{z}$, then $c_i = 0$ for *each* i $(1 \le i \le m)$.

We now consider some examples.

Example 15.18 *Determine whether* $S = \{(1,1,1),(1,1,0),(0,1,1)\}$ *is a linearly independent set of vectors in* $\mathbf{R}^3$.

Solution. Let a, b, and c be scalars such that

$$a \cdot (1,1,1) + b \cdot (1,1,0) + c \cdot (0,1,1) = (0,0,0).$$

By scalar multiplication and vector addition, we have $(a+b, a+b+c, a+c) = (0,0,0)$, arriving at the following system of equations:

$$\begin{aligned} a+b &= 0 \\ a+b+c &= 0 \\ a+c &= 0. \end{aligned}$$

Subtracting the first equation from the second, we obtain $c = 0$. Substituting $c = 0$ into the third equation, we obtain $a = 0$. Substituting $a = 0$ and $c = 0$ into the second equation, we obtain $b = 0$. Hence $a = b = c = 0$ and S is linearly independent. $\diamondsuit$

Example 15.19 *Determine whether*

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

is a linearly independent set of vectors in $M_2(\mathbf{R})$.

Solution. Again, let a, b, and c be scalars such that

$$a \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

By scalar multiplication and matrix addition, we have

$$\begin{bmatrix} 2a+c & a+b+c \\ a+b+c & 2b+c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This results in the system of equations:

$$\begin{aligned} 2a+c &= 0 \\ a+b+c &= 0 \\ 2b+c &= 0 \end{aligned}$$

where the second equation actually occurs twice. From the first and third equations, it follows that $c = -2a$ and $c = -2b$ and so $a = b = -c/2$. Substituting these values for a

and b in the second equation gives $(-c/2) + (-c/2) + c = -c + c = 0$, that is, the second equation is satisfied for every value of c. Hence, if we let $c = -2$, say, then $a = b = 1$ and

$$1 \cdot \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Consequently, S is a linearly dependent set of vectors. $\diamondsuit$

We now show that a familiar set of polynomial functions is linearly independent.

Theorem to Prove For every nonnegative integer n, the set $S_n = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathbf{R}[x]$.

Proof Strategy The elements of S_n are actually functions, say $S_n = \{f_0, f_1, f_2, \ldots, f_n\}$, where $f_i : \mathbf{R} \to \mathbf{R}$ is defined by $f_i(x) = x^i$ for $0 \le i \le n$ and for all $x \in \mathbf{R}$. To show that S_n is linearly independent, we are required to show that if $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$, where $c_i \in \mathbf{R}$ for $0 \le i \le n$, then $c_i = 0$ for all i. Of course, the question is how to do this. By choosing various values of x, we could arrive at a system of equations to solve. For example, we could begin by letting $x = 0$, obtaining $c_0 \cdot 1 + c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0$, and so $c_0 = 0$. Therefore, $c_1 x + c_2 x^2 + \ldots + c_n x^n = 0$. Letting $x = 1$ and $x = 2$, we have $c_1 + c_2 + \ldots + c_n = 0$ and $2c_1 + 2^2 c_2 + \ldots + 2^n c_n = 0$. We could actually arrive at a system of n equations and n unknowns, but perhaps this is sounding complicated.

On the other hand, from the statement of the theorem, another approach is suggested. Quite often when we see a theorem stated as "for every nonnegative integer n", we think of applying induction. The main challenge to such a proof would be to show that if $\{1, x, x^2, \ldots, x^k\}$ is linearly independent, where $k \ge 0$, then $\{1, x, x^2, \ldots, x^{k+1}\}$ is linearly independent. Hence we would be dealing with the equation $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1} = 0$ for $c_i \in \mathbf{R}$, $0 \le i \le k+1$, attempting to show that $c_i = 0$ for all i $(0 \le i \le k+1)$. We already mentioned that showing $c_0 = 0$ is not difficult. In order to make use of the induction hypothesis, we need a linear combination of the polynomials $1, x, x^2, \ldots, x^k$. One idea for doing this is to take the derivative of $c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1}$. $\diamondsuit$

Theorem 15.20 *For every nonnegative integer n, the set $S_n = \{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathbf{R}[x]$.*

Proof. We proceed by induction. For $n = 0$, we are required to show that $S_0 = \{1\}$ is linearly independent in $\mathbf{R}[x]$. Let c be a scalar such that $c \cdot 1 = 0$. Then surely $c = 0$ and so S_0 is linearly independent.

Assume that $S_k = \{1, x, x^2, \ldots, x^k\}$ is linearly independent in $\mathbf{R}[x]$, where k is a nonnegative integer. We show that $S_{k+1} = \{1, x, x^2, \ldots, x^{k+1}\}$ is linearly independent in $\mathbf{R}[x]$. Let $c_0, c_1, \ldots, c_{k+1}$ be scalars such that

$$c_0 \cdot 1 + c_1 x + c_2 x^2 + \ldots + c_{k+1} x^{k+1} = 0, \tag{15.2}$$

for all $x \in \mathbf{R}$. Letting $x = 0$ in (15.2), we see that $c_0 = 0$. Now taking the derivatives of both sides of (15.2), we see that

$$c_1 \cdot 1 + 2c_2 x + 3c_3 x^2 + \ldots + (k+1) c_{k+1} x^k = 0$$

for all $x \in \mathbf{R}$. By the induction hypothesis, S_k is a linearly independent set of vectors in $\mathbf{R}[x]$ and so $c_1 = 2c_2 = 3c_3 = \ldots = (k+1)c_{k+1} = 0$, which implies that $c_1 = c_2 = c_3 = \ldots = c_{k+1} = 0$. Since $c_0 = 0$ as well, it follows that S_{k+1} is linearly independent. ■

Proof Analysis Before proceeding further, it is important that we understand the proof we have just given. The proof began by showing that $S_0 = \{1\}$ is linearly independent. What this means is that S_0 consists of the single constant polynomial function f defined by $f(x) = 1$ for all $x \in \mathbf{R}$. Let c be a scalar (real number) such that $c \cdot f = f_0$, where f_0 is the zero polynomial function defined by $f_0(x) = 0$ for all $x \in \mathbf{R}$. Thus, for each $x \in \mathbf{R}$, $(cf)(x) = f_0(x) = 0$, that is,

$$(cf)(x) = c \cdot f(x) = c \cdot 1 = 0 = f_0(x)$$

and so $c = 0$. $\diamondsuit$

We now consider a result for a general vector space.

Result 15.21 *If $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ are linearly independent vectors in a vector space V, then $\mathbf{v}_1$, $\mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are also linearly independent in V.*

Proof. Let $a, b,$ and c be scalars such that

$$a \cdot \mathbf{v}_1 + b \cdot (\mathbf{v}_1 + \mathbf{v}_2) + c \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = \mathbf{z}.$$

From this, we have

$$(a + b + c) \cdot \mathbf{v}_1 + (b + c) \cdot \mathbf{v}_2 + c \cdot \mathbf{v}_3 = \mathbf{z}.$$

Since $\mathbf{v}_1, \mathbf{v}_2,$ and $\mathbf{v}_3$ are linearly independent, $a + b + c = b + c = c = 0$, from which it follows that $a = b = c = 0$ and so $\mathbf{v}_1$, $\mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$ are linearly independent. ■

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n vectors, where $n \in \mathbf{N}$, and let S' be a nonempty subset of S. Then $|S'| = m$ for some integer m with $1 \le m \le n$. Since the order in which the elements of S are listed is irrelevant, these elements can be rearranged and relabeled if necessary so that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$. This fact is quite useful at times.

Theorem 15.22 *Let S be a finite nonempty set of vectors in a vector space V. If S is linearly independent in V and S' is a nonempty subset of S, then S' is also linearly independent in V.*

Proof. We may assume that $S' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ and $S = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n\}$, where then $1 \le m \le n$. If $m = n$, then $S' = S$ and surely S' is linearly independent. Thus we can assume that $m < n$. Let $c_1, c_2, \ldots, c_m$ be scalars such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{z}.$$

However, then,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + 0\mathbf{v}_{m+2} + \ldots + 0\mathbf{v}_n = \mathbf{z}. \tag{15.3}$$

Since S is linearly independent, all scalars in (15.3) are 0. In particular, $c_1 = c_2 = \ldots = c_m = 0$, which implies that S' is linearly independent. ■

We can restate Theorem 15.22 as follows: Let V be a vector space, and let S and S' be finite nonempty subsets of V such that $S' \subseteq S$. If S is linearly independent, then S' is linearly independent. The contrapositive of this implication gives us: If S' is linearly dependent, then S is linearly dependent.

Although we have only discussed linear independence and linear dependence in connection with finite sets of vectors, these concepts exist for infinite sets of vectors as well. An infinite set of vectors in a vector space V is **linearly independent** if *every* finite nonempty subset of S is linearly independent. Equivalently, an infinite set S of vectors in a vector space V is **linearly dependent** if some finite nonempty subset of S is linearly dependent. Every example we have seen of a (finite) set S of linearly dependent vectors in some vector space V gives rise to an infinite set T of linearly dependent vectors; namely, any infinite subset T of V such that $S \subseteq T$ is linearly dependent. But what is an example of a vector space that contains infinitely many linearly independent vectors? We provide such an example now.

Result 15.23 *The set $T = \{1, x, x^2, \ldots\}$ is linearly independent in $\mathbf{R}[x]$.*

Proof. Let S be a finite nonempty subset of T. Then there is a largest nonnegative integer m such that $x^m \in S$. Therefore, $S \subseteq S_m = \{1, x, x^2, \ldots, x^m\}$. By Theorem 15.20, S_m is linearly independent in $\mathbf{R}[x]$ and by Theorem 15.22, S is linearly independent. Consequently, T is linearly independent in $\mathbf{R}[x]$. ■

15.8 Linear Transformations

We have seen that many properties of a vector space V, subspaces of V, the span of a set of vectors in V, and linear independence and linear dependence of vectors in V deal with a common concept: linear combinations of vectors. Perhaps this is not unexpected in an area of mathematics called linear algebra. There are occasions when two vectors spaces V and V' are so closely linked that with each vector $\mathbf{w} \in V$, there is an associated vector $\mathbf{w}' \in V'$ such that the vector associated with $\alpha\mathbf{u} + \beta\mathbf{v}$ in V is $\alpha\mathbf{u}' + \beta\mathbf{v}'$ in V'. Such an association describes a function from V to V'. In particular, a function $f : V \to V'$ is said to **preserve linear combinations** of vectors if $f(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$ and every two scalars α and β. If $f : V \to V'$ has the property that $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$, then f is said to **preserve addition**; while if $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$ for all $\mathbf{u} \in V$ and every scalar α, then f is said to **preserve scalar multiplication**.

Let $\mathbf{z}'$ be the zero vector of V'. If $f : V \to V'$ preserves linear combinations and $\mathbf{u}, \mathbf{v} \in V$, then

$$f(\mathbf{u} + \mathbf{v}) = f(1 \cdot \mathbf{u} + 1 \cdot \mathbf{v}) = 1 \cdot f(\mathbf{u}) + 1 \cdot f(\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$

and $f(\alpha\mathbf{u}) = f(\alpha\mathbf{u} + 0\mathbf{v}) = \alpha f(\mathbf{u}) + 0f(\mathbf{v}) = \alpha f(\mathbf{u}) + \mathbf{z}' = \alpha f(\mathbf{u})$. Hence if $f : V \to V'$ is a function that preserves linear combinations, then f preserves addition and scalar multiplication as well.

Conversely, suppose that $f : V \to V'$ is a function that preserves both addition and scalar multiplication. Then for $\mathbf{u}, \mathbf{v} \in V$ and scalars α and β,

$$f(\alpha\mathbf{u} + \beta\mathbf{v}) = f(\alpha\mathbf{u}) + f(\beta\mathbf{v}) = \alpha f(\mathbf{u}) + \beta f(\mathbf{v}),$$

that is, f preserves linear combinations. Because functions that preserve linear combinations are so important in linear algebra, they are given a special name.

Let V and V' be vector spaces. A function $T : V \to V'$ is called a **linear transformation** if it preserves both addition and scalar multiplication, that is, if it satisfies the following conditions:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$

for all $\mathbf{u}, \mathbf{v} \in V$ and all $\alpha \in \mathbf{R}$. There are some points in connection with these conditions that need to be addressed and that may not be self-evident. Condition 1 states that $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every two vectors $\mathbf{u}$ and $\mathbf{v}$ of V. Hence the addition indicated in $T(\mathbf{u}+\mathbf{v})$ takes place in V; while, on the other hand, since $T(\mathbf{u})$ and $T(\mathbf{v})$ are vectors in V', the addition indicated in $T(\mathbf{u}) + T(\mathbf{v})$ takes place in V'. Also, condition 2 states that $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$ for every vector $\mathbf{v}$ in V and every scalar α. By the same reasoning, the scalar multiplication indicated in $T(\alpha\mathbf{v})$ takes place in V, while the scalar multiplication in $\alpha T(\mathbf{v})$ takes place in V'. From what we have already seen, every linear transformation preserves linear combinations of vectors (hence the name).

Let's consider an example of a linear transformation.

Result 15.24 *The function* $T : \mathbf{R}^3 \to \mathbf{R}^2$ *defined by*

$$T((a,b,c)) = T(a,b,c) = (2a+c, 3c-b)$$

is a linear transformation.

Before we prove Result 15.24, let's be certain that we understand what this function does. For example, $T(1,2,3) = (5,7)$, $T(1,-6,-2) = (0,0)$, while $T(0,0,0) = (0,0)$ as well. We now show that T is a linear transformation.

Proof of Result 15.24. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$. Then $\mathbf{u} = (a,b,c)$ and $\mathbf{v} = (d,e,f)$ for $a,b,c,d,e,f \in \mathbf{R}$. Then

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(a+d, b+e, c+f) = (2(a+d)+c+f, 3(c+f)-(b+e)) \\ &= (2a+c, 3c-b) + (2d+f, 3f-e) \\ &= T(a,b,c) + T(d,e,f) = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} T(\alpha\mathbf{u}) &= T(\alpha(a,b,c)) = T(\alpha a, \alpha b, \alpha c) \\ &= (2\alpha a + \alpha c, 3\alpha c - \alpha b) = \alpha(2a+c, 3c-b) = \alpha T(\mathbf{u}), \end{aligned}$$

as desired. ■

Sometimes the vectors in $\mathbf{R}^3$ are written as "column vectors", that is, as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ rather than (a,b,c) or the "row vector" $[a\ b\ c]$. In this case, notice that the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(a,b,c) = (2a+c, 3c-b)$ can be described as

$$T(a,b,c) = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+c \\ -b+3c \end{bmatrix},$$

that is, if we let $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \end{bmatrix}$, then this linear transformation can be defined in terms of the matrix A, namely,

$$T(\mathbf{v}) = A\mathbf{v}.$$

In general, if A is an $m \times n$ matrix, then the function $T : \mathbf{R}^n \to \mathbf{R}^m$ defined by $T(\mathbf{u}) = A\mathbf{u}$ for an $n \times 1$ column vector $\mathbf{u} \in \mathbf{R}^n$ is a linear transformation. For example, consider the 3×2 matrix $A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}$. For $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, and $\alpha \in \mathbf{R}$,

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} a+c \\ b+d \end{bmatrix} = \begin{bmatrix} a+c-2b-2d \\ 3a+3c-b-d \\ 2a+2c+5b+5d \end{bmatrix} \\ &= \begin{bmatrix} a-2b \\ 3a-b \\ 2a+5b \end{bmatrix} + \begin{bmatrix} c-2d \\ 3c-d \\ 2c+5d \end{bmatrix} = T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \\ &= T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} T(\alpha\mathbf{u}) &= T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = \begin{bmatrix} 1 & -2 \\ 3 & -1 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix} = \begin{bmatrix} \alpha a - 2\alpha b \\ 3\alpha a - \alpha b \\ 2\alpha a + 5\alpha b \end{bmatrix} \\ &= \alpha\begin{bmatrix} a-2b \\ 3a-b \\ 2a+5b \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \alpha T(\mathbf{u}). \end{aligned}$$

Thus, $T : \mathbf{R}_2 \to \mathbf{R}_3$ is a linear transformation. The proof for a general $m \times n$ matrix is similar. As another illustration of a linear transformation, we consider a well-known function from $\mathbf{R}[x]$ to itself.

Result 15.25 *The function D (for differentiation) from $\mathbf{R}[x]$ to $\mathbf{R}[x]$ defined by*

$$D(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n) = c_1 + 2c_2x + \ldots + nc_nx^{n-1}$$

is a linear transformation.

Proof. Let $f, g \in \mathbf{R}[x]$, where $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_rx^r$ and $g(x) = b_0 + b_1x + b_2x^2 + \ldots + b_sx^s$ and, say, $r \le s$. Then

$$\begin{aligned} D(f(x)+g(x)) &= D\left((a_0 + a_1x + \ldots + a_rx^r) + (b_0 + b_1x + \ldots + b_sx^s)\right) \\ &= D\Big((a_0+b_0) + (a_1+b_1)x + \ldots + (a_r+b_r)x^r + b_{r+1}x^{r+1} + \ldots + b_sx^s\Big) \\ &= (a_1+b_1) + \ldots + r(a_r+b_r)x^{r-1} + (r+1)b_{r+1}x^r + \ldots + sb_sx^{s-1} \\ &= \Big(a_1 + 2a_2x + \ldots + ra_rx^{r-1}\Big) + \Big(b_1 + 2b_2x + \ldots + sb_sx^{s-1}\Big) \\ &= D(f(x)) + D(g(x)) \end{aligned}$$

and

$$\begin{aligned} D(\alpha f(x)) &= D\Big(\alpha a_0 + \alpha a_1x + \alpha a_2x^2 + \ldots + \alpha a_rx^r\Big) \\ &= \alpha a_1 + 2\alpha a_2x + \ldots + r\alpha a_rx^{r-1} \\ &= \alpha(a_1 + 2a_2x + \ldots + ra_rx^{r-1}) = \alpha D(f(x)). \end{aligned}$$

Since D preserves both addition and scalar multiplication, it is a linear transformation. ∎

There is a special kind a function from a vector space to itself that is always a linear transformation.

Result 15.26 *Let V be a vector space over the set $\mathbf{R}$ of real numbers. For $c \in \mathbf{R}$, the function $T : V \to V$ defined by $T(\mathbf{v}) = c\mathbf{v}$ is a linear transformation.*

Proof. Let $\mathbf{u}, \mathbf{w} \in V$. Then

$$T(\mathbf{u} + \mathbf{w}) = c(\mathbf{u} + \mathbf{w}) = c\mathbf{u} + c\mathbf{w} = T(\mathbf{u}) + T(\mathbf{w});$$

while, for $\alpha \in \mathbf{R}$,

$$T(\alpha\mathbf{u}) = c(\alpha\mathbf{u}) = (c\alpha)(\mathbf{u}) = (\alpha c)(\mathbf{u}) = \alpha(c\mathbf{u}) = \alpha T(\mathbf{u}).$$

Therefore, T is a linear transformation. ■

For $c = 1$, the function T defined in Result 15.26 is the identity function; while for $c = 0$, the function T maps every vector into the zero vector. Consequently, both of these functions are linear transformations.

We now look at functions involving other vector spaces. For a function $f \in \mathcal{F}_{\mathbf{R}}$ and a real number r, we define the function $f + r$ by $(f + r)(x) = f(x) + r$ for all $x \in \mathbf{R}$.

Example 15.27 *Let r be a nonzero real number. Prove or disprove: The function $T : \mathcal{F}_{\mathbf{R}} \to \mathcal{F}_{\mathbf{R}}$ defined by $T(f) = f + r$ is a linear transformation.*

Solution. Let $f, g \in \mathcal{F}_{\mathbf{R}}$. Observe that

$$T(f + g) = (f + g) + r,$$

while

$$T(f) + T(g) = (f + r) + (g + r) = (f + g) + 2r.$$

Since $r \neq 0$, it follows that $T(f + g) \neq T(f) + T(g)$. Therefore, T is not a linear transformation. ♢

Example 15.28 *Let $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ be a function defined by*

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} ad & 0 \\ 0 & bc \end{bmatrix}.$$

Prove or disprove: T is a linear transformation.

Solution. Since

$$T\left(2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

and

$$2T\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

T is not a linear transformation. ♢

Example 15.29 *The function $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ is defined by*

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}.$$

Prove or disprove: T is a linear transformation.

Solution. Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbf{R})$ and $\alpha \in \mathbf{R}$. Then

$$\begin{aligned} T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2 & a_1+a_2 \\ c_1+c_2 & c_1+c_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 \\ c_2 & c_2 \end{bmatrix} \\ &= T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right); \end{aligned}$$

while

$$\begin{aligned} T\left(\alpha\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}\right) = \begin{bmatrix} \alpha a_1 & \alpha a_1 \\ \alpha c_1 & \alpha c_1 \end{bmatrix} \\ &= \alpha\begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix} = \alpha T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right). \end{aligned}$$

Since T preserves both addition and scalar multiplication, T is a linear transformation. ♢

15.9 Properties of Linear Transformations

An important property of linear transformations is that the composition of any two linear transformations (when the composition is defined) is also a linear transformation. This fact has an interesting consequence as well.

Theorem 15.30 *Let V, V', and V'' be vector spaces. If $T_1 : V \to V'$ and $T_2 : V' \to V''$ are linear transformations, then the composition $T_2 \circ T_1 : V \to V''$ is a linear transformation as well.*

Proof. For $\mathbf{u}, \mathbf{v} \in V$ and a scalar α, observe that

$$\begin{aligned} (T_2 \circ T_1)(\mathbf{u}+\mathbf{v}) &= T_2(T_1(\mathbf{u}+\mathbf{v})) = T_2(T_1(\mathbf{u}) + T_1(\mathbf{v})) \\ &= T_2(T_1(\mathbf{u})) + T_2(T_1(\mathbf{v})) = (T_2 \circ T_1)(\mathbf{u}) + (T_2 \circ T_1)(\mathbf{v}) \end{aligned}$$

and

$$\begin{aligned} (T_2 \circ T_1)(\alpha\mathbf{v}) &= T_2(T_1(\alpha\mathbf{v})) = T_2(\alpha T_1(\mathbf{v})) \\ &= \alpha T_2(T_1(\mathbf{v})) = \alpha(T_2 \circ T_1)(\mathbf{v}). \end{aligned}$$

Therefore, $T_2 \circ T_1$ is a linear transformation. ■

As an example of the preceding theorem, let $T_1 : \mathbf{R}^3 \to \mathbf{R}^2$ and $T_2 : \mathbf{R}^2 \to \mathbf{R}^3$ be defined by $T_1(a,b,c) = (a+2b-c, 3b+2c)$ and $T_2(a,b) = (b, 2a, a+b)$. Then $T_2 \circ T_1 : \mathbf{R}^3 \to \mathbf{R}^3$ is given by

$$\begin{aligned} (T_2 \circ T_1)(a,b,c) &= T_2(T_1(a,b,c)) \\ &= T_2(a+2b-c, 3b+2c) \\ &= (3b+2c, 2a+4b-2c, a+5b+c). \end{aligned}$$

From what we mentioned earlier, T_1 and T_2 can also be defined by

$$T_1\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } T_2\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}.$$

Interestingly enough,

$$(T_2 \circ T_1)\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

that is, the composition $T_2 \circ T_1$ can be obtained by multiplying the matrices that describe T_1 and T_2. Therefore, if we represent the linear transformations T_1 and T_2 by matrices A_1 and A_2, respectively, then the matrix that represents $T_2 \circ T_1$ is A_2A_1. This also explains why the definition of matrix multiplication, though curious at first, is actually quite logical.

Two fundamental properties of a linear transformation are given in the next theorem.

Theorem 15.31 *Let V and V' be vector spaces with respective zero vectors $\mathbf{z}$ and $\mathbf{z}'$. If $T : V \to V'$ is a linear transformation, then*

(*i*) $T(\mathbf{z}) = \mathbf{z}'$ *and*

(*ii*) $T(-\mathbf{v}) = -T(\mathbf{v})$ *for all* $\mathbf{v} \in V$.

Proof. We first verify (*i*). Since T preserves scalar multiplication,

$$T(\mathbf{z}) = T(0\mathbf{z}) = 0T(\mathbf{z}) = \mathbf{z}'.$$

Next we verify (*ii*). Let $\mathbf{v} \in V$. Then

$$T(\mathbf{v}) + T(-\mathbf{v}) = T(\mathbf{v} + (-\mathbf{v})) = T(\mathbf{z}) = \mathbf{z}',$$

the last equality following by (*i*). Since the vector $T(\mathbf{v})$ in V' has a unique negative, namely $-T(\mathbf{v})$, we conclude that $T(-\mathbf{v}) = -T(\mathbf{v})$. ■

If $T : V \to V'$ is a linear transformation, then it is often of interest to know how T acts on subspaces of V. Let's recall some terminology and notation from functions. In a linear transformation $T : V \to V'$, the set V is the **domain** of T and the set V' is the **codomain** of T. If W is a subset of V, then $T(W) = \{T(\mathbf{w}) : \mathbf{w} \in W\}$ is the **image** of W under T. In particular, $T(V)$ is the **range** of T.

Theorem 15.32 *Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. If W is a subspace of V, then $T(W)$ is a subspace of V'.*

Proof. Let $\mathbf{z}$ and $\mathbf{z}'$ be the zero vectors in V and V', respectively. Since $\mathbf{z} \in W$ and $T(\mathbf{z}) = \mathbf{z}'$ by Theorem 15.31, it follows that $\mathbf{z}' \in T(W)$ and so $T(W) \neq \emptyset$. Thus we need only show that $T(W)$ is closed under addition and scalar multiplication. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $T(W)$. Hence, there exist vectors $\mathbf{u}$ and $\mathbf{v}$ in W such that $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y}$. Then

$$\mathbf{x} + \mathbf{y} = T(\mathbf{u}) + T(\mathbf{v}) = T(\mathbf{u} + \mathbf{v}).$$

Since $\mathbf{u}, \mathbf{v} \in W$ and W is a subspace of V, it follows that $\mathbf{u} + \mathbf{v} \in W$. Hence $\mathbf{x} + \mathbf{y} = T(\mathbf{u} + \mathbf{v}) \in T(W)$.

Next let α be a scalar and $\mathbf{x} \in T(W)$. We show that $\alpha\mathbf{x} \in T(W)$. Since $\mathbf{x} \in T(W)$, there exists $\mathbf{u} \in W$ such that $T(\mathbf{u}) = \mathbf{x}$. Now

$$\alpha\mathbf{x} = \alpha T(\mathbf{u}) = T(\alpha\mathbf{u}).$$

Since $\alpha\mathbf{u} \in W$, it follows that $\alpha\mathbf{x} = T(\alpha\mathbf{u}) \in T(W)$. By the Subspace Test, $T(W)$ is a subspace of V'. ∎

To illustrate Theorem 15.32, let's return to the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined in Result 15.24 by $T(a, b, c) = (2a + c, 3c - b)$. Let $W = \{(a, b, 0) \ : \ a, b \in \mathbf{R}\}$. We use the Subspace Test to show that W is a subspace of $\mathbf{R}^3$. Since $(0, 0, 0) \in W$, it follows that $W \neq \emptyset$. Let $(a_1, b_1, 0), (a_2, b_2, 0) \in W$ and let $\alpha \in \mathbf{R}$. Then

$$(a_1, b_1, 0) + (a_2, b_2, 0) = (a_1 + a_2, b_1 + b_2, 0) \in W \text{ and } \alpha(a_1, b_1, 0) = (\alpha a_1, \alpha b_1, 0) \in W.$$

Since W is closed under addition and scalar multiplication, W is a subspace of $\mathbf{R}^3$. By Theorem 15.32, $T(W) = \{(2a, -b) \ : \ a, b \in \mathbf{R}\}$ is a a subspace of $\mathbf{R}^2$. We show in fact that $T(W) = \mathbf{R}^2$. Certainly, $\mathbf{R}^2 = \langle (1, 0), (0, 1) \rangle$. Hence to show that $T(W) = \mathbf{R}^2$, it suffices, by Corollary 15.16, to show that $(1, 0)$ and $(0, 1)$ belong to $T(W)$. Letting $a = 1/2$ and $b = 0$, we see that $(1, 0) \in T(W)$; while letting $a = 0$ and $b = -1$, we see that $(0, 1) \in T(W)$.

For this same linear transformation T, we saw that $T(1, -6, -2) = (0, 0)$ and $T(0, 0, 0) = (0, 0)$. Hence both $(1, -6, -2)$ and $(0, 0, 0)$ map into the zero vector of $\mathbf{R}^2$. The fact that $(0, 0, 0)$ maps into $(0, 0)$ is not surprising, of course, since Theorem 15.31 guarantees this.

If $T : V \to V'$ is a linear transformation and W' is a subset of V', then

$$T^{-1}(W') = \{\mathbf{v} \in V \ : \ T(\mathbf{v}) \in W'\}$$

is called the **inverse image** of W' under T. If $W' = \{\mathbf{z}'\}$, where $\mathbf{z}'$ is the zero vector of V', then $T^{-1}(W')$ is called the **kernel** of T and is denoted by $\ker(T)$. That is, the kernel of $T : V \to V'$ is the set

$$\ker(T) = T^{-1}(\{\mathbf{z}'\}) = \{v \in V \ : \ T(v) = \mathbf{z}'\}.$$

An interesting feature of the kernel lies in the following theorem.

Theorem 15.33 *Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. Then the kernel of T is a subspace of V.*

Proof. Let $\mathbf{z}$ and $\mathbf{z}'$ be the zero vectors of V and V', respectively. Since $T(\mathbf{z}) = \mathbf{z}'$, it follows that $\mathbf{z} \in \ker(T)$ and so $\ker(T) \neq \emptyset$. Now let $\mathbf{u}, \mathbf{v} \in \ker(T)$ and $\alpha \in \mathbf{R}$. Then

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(v) = \mathbf{z}' + \mathbf{z}' = \mathbf{z}'$$

and

$$T(\alpha\mathbf{u}) = \alpha T(\mathbf{u}) = \alpha\mathbf{z}' = \mathbf{z}'.$$

This implies that $\mathbf{u} + \mathbf{v} \in \ker(T)$ and $\alpha\mathbf{u} \in \ker(T)$. By the Subspace Test, $\ker(T)$ is a subspace of V. ∎

Returning once again to the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ in Result 15.24 defined by $T(a, b, c) = (2a + c, 3c - b)$, we see that

$$\ker(T) = \{(a,b,c) \ : \ 2a + c = 0 \text{ and } 3c - b = 0\}$$

is a subspace of $\mathbf{R}^3$. Since $2a + c = 0$ and $3c - b = 0$, it follows that $a = -c/2$ and $b = 3c$. Thus $\ker(T) = \{(-c/2, 3c, c) \ : \ c \in \mathbf{R}\}$. In other words, $\ker(T)$ is the subspace of $\mathbf{R}^3$ consisting of all scalar multiples of $(-1/2, 3, 1)$.

Exercises for Chapter 15

15.1 Prove that the set $\mathcal{C} = \{a + bi \ : \ a, b \in \mathbf{R}\}$ of complex numbers is a vector space under the addition $(a + bi) + (c + di) = (a + c) + (b + d)i$ and scalar multiplication $\alpha(a + bi) = \alpha a + \alpha bi$, where $\alpha \in \mathbf{R}$.

15.2 Although we have taken $\mathbf{R}$ to be the set of scalars in a vector space, this need not always be the case. Let $V = \{([a], [b]) \ : \ [a], [b] \in \mathbf{Z}_3\}$ and let $\mathbf{Z}_3$ be the set of scalars.

(a) Show that V is a vector space over the set $\mathbf{Z}_3$ of scalars under the addition $([a], [b]) + ([c], [d]) = ([a + c], [b + d])$ and scalar multiplication $[c]([a], [b]) = ([ca], [cb])$.

(b) Write out precisely the elements of V. (Hence a vector space can have more than one vector and be finite.)

15.3 Addition or scalar multiplication is defined in $\mathbf{R}^3$ in each of the following. (Each operation not defined is taken as the standard one.) Under these operations, determine whether $\mathbf{R}^3$ is a vector space.

(a) $(a, b, c) + (d, e, f) = (a, b, c)$

(b) $(a, b, c) + (d, e, f) = (a - d, b - e, c - f)$

(c) $(a, b, c) + (d, e, f) = (0, 0, 0)$

(d) $\alpha(a, b, c) = (a, b, c)$

(e) $\alpha(a, b, c) = (b, c, a)$

(f) $\alpha(a, b, c) = (0, 0, 0)$

(g) $\alpha(a, b, c) = (\alpha a, 3\alpha b, \alpha c)$

15.4 Let V be a vector space, where $\mathbf{u}, \mathbf{v} \in V$. Prove that there exists a unique vector $\mathbf{x}$ in V such that $\mathbf{u} + \mathbf{x} = \mathbf{v}$.

15.5 Let V be a vector space with $\mathbf{v} \in V$ and $\alpha \in \mathbf{R}$. Prove that $\alpha(-\mathbf{v}) = (-\alpha)\mathbf{v} = -(\alpha\mathbf{v})$.

15.6 (a) Let V be a vector space and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Prove that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$. (This is the cancellation property for addition of vectors.)

(b) Use (a) to prove Theorem 15.3.

15.7 Prove or disprove:

(a) No vector is its own negative.

(b) Every vector is the negative of some vector.

(c) Every vector space has at least two vectors.

15.8 Let V be a vector space containing nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Prove that if $\mathbf{u} \neq \alpha\mathbf{v}$ for each $\alpha \in \mathbf{R}$, then $\mathbf{u} \neq \beta(\mathbf{u}+\mathbf{v})$ for each $\beta \in \mathbf{R}$.

15.9 Determine which of following subsets of $\mathbf{R}^4$ are subspaces of $\mathbf{R}^4$.

(a) $W_1 = \{(a, a, a, a) \ : \ a \in \mathbf{R}\}$

(b) $W_2 = \{(a, 2b, 3a, 4b) \ : \ a, b \in \mathbf{R}\}$

(c) $W_3 = \{(a, 0, 0, 1) \ : \ a \in \mathbf{R}\}$

(d) $W_4 = \{(a, a^2, 0, 0) \ : \ a \in \mathbf{R}\}$

(e) $W_5 = \{(a, b, a+b, b) \ : \ a, b \in \mathbf{R}\}$

15.10 Let $\mathcal{F}_\mathbf{R}$ be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Determine which of the following subsets of $\mathcal{F}_\mathbf{R}$ are subspaces of $\mathcal{F}_\mathbf{R}$.

(a) W_1 consists of all functions f such that $f(1) = 0 = f(2)$.

(b) W_2 consists of all functions f such that $f(1) = 0$ or $f(2) = 0$.

(c) W_3 consists of all functions f such that $f(2) = 2f(1)$.

(d) W_4 consists of all functions f such that $f(1) \neq f(2)$.

(d) W_5 consists of all functions f such that $f(1) \neq 0$.

15.11 Recall that the set $\mathbf{R}[x]$ of polynomial functions is a subspace of $\mathcal{F}_\mathbf{R}$. Now determine which of the following subsets of $\mathbf{R}[x]$ are subspaces of $\mathbf{R}[x]$.

(a) $U_1 = \{f \ : \ f(x) = a$ for a fixed real number $a\}$ (The set of all constant polynomials)

(b) $U_2 = \{f \ : \ f(x) = a + bx + cx^2 + dx^3, \ a, b, c, d \in \mathbf{R}, d \neq 0\}$

(c) $U_3 = \{f \ : \ f(x) = a + bx + cx^2 + dx^3, a, b, c, d \in \mathbf{R}\}$

(d) $U_4 = \{f \ : \ f(x) = a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2m}x^{2m}, m \geq 0,$ and $a_i \in \mathbf{R}$ for $0 \leq i \leq m\}$

(e) $U_5 = \{f \ : \ f(x) = (x^3+1)g(x)$ for some $g \in \mathbf{R}[x]\}$

15.12 Let $M_2(\mathbf{R})$ be the vector space of 2×2 matrices whose entries are real numbers. Determine which of the following subsets of $M_2(\mathbf{R})$ are subspaces of $M_2(\mathbf{R})$.

(a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ ad - bc = 0 \right\}$

(b) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d = 0 \right\}$, where α_1, α_2, α_3, α_4 are fixed real numbers.

15.13 Prove that

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} \ : \ a_i \in \mathbf{R} \text{ for } 1 \leq i \leq 6 \right\}$$

is a subspace of the vector space $M_3[\mathbf{R}]$.

15.14 Let U and W be subspaces of a vector space V. Prove that $U \cap W$ is a subspace of V.

15.15 The graph of the function$f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \frac{3}{5}x$ is a straight line in $\mathbf{R}^2$ passing through the origin. Each point (x, y) on this graph is a solution of the equation $3x - 5y = 0$. Prove that the set S of solutions of this equation is a subspace of $\mathbf{R}^2$.

15.16 Determine the following linear combinations:

(a) $4 \cdot (1, -2, 3) + (-2) \cdot (1, -1, 0)$

(b) $(-1) \begin{bmatrix} 3 & -2 \\ 1 & -3 \end{bmatrix} + 2 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + 5 \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

15.17 In $\mathbf{R}^3$, write $\mathbf{i} = (1, 0, 0)$ as a linear combination of $\mathbf{u}_1 = (0, 1, 1), \mathbf{u}_2 = (1, 0, 1)$, and $\mathbf{u}_3 = (1, 1, 0)$.

15.18 Let $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (0, 1, 2)$, and $\mathbf{w} = (3, 1, -1)$ be vectors in $\mathbf{R}^3$.

(a) Show that $\mathbf{w}$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

(b) Show that the vector $\mathbf{x} = (8, 5, 2)$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in more than one way.

15.19 Let V be a vector space containing the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\ldots$, $\mathbf{w}_m$. Let $W = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$ and $W' = \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \rangle$. Prove that if each vector v_i $(1 \leq i \leq n)$ is a linear combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$, then $W \subseteq W'$.

15.20 Prove that $\langle (1, 2, 3), (0, 4, 1) \rangle = \langle (1, 6, 4), (1, -2, 2) \rangle$ in $\mathbf{R}^3$

15.21 Let V be a vector space and let $\mathbf{u}$ and $\mathbf{v}$ in V. Prove that

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, 2\mathbf{u} + \mathbf{v} \rangle$

(b) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle$

15.22 Determine which sets S of vectors are linearly independent in the indicated vector space V.

(a) $S = \{(1, 1, 1), (1, -2, 3), (2, 5, -1)\}; V = \mathbf{R}^3$.

(b) $S = \{(1, 0, -1), (2, 1, 1), (0, 1, 3)\}; V = \mathbf{R}^3$.

(c) $S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}; V = M_2(\mathbf{R})$.

15.23 For the vectors $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 0, 2)$, find a vector $\mathbf{w}$ such that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent in $\mathbf{R}^3$. Verify that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

15.24 Prove or disprove: If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent vectors in a vector space V, then $\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_3, 2\mathbf{u}_3$ are linearly independent vectors in V.

15.25 Determine which sets S of vectors in $\mathcal{F}_{\mathbf{R}}$ are linearly independent.

(a) $S = \{1, \sin^2 x, \cos^2 x\}$

(b) $S = \{1, \sin x, \cos x\}$

(c) $S = \{1, e^x, e^{-x}\}$

(d) $S = \{1, x, x/(x^2+1)\}$.

15.26 Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a linearly dependent set of $n \geq 2$ vectors in a vector space V. Prove that if each subset of S consisting of $n-1$ vectors is linearly independent, then there exist nonzero scalars $c_1, c_2, \ldots, c_n$ such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = \mathbf{z}$.

15.27 Prove that if $T : V \to V'$ is a linear transformation, then

$$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \ldots + \alpha_n\mathbf{v}_n) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \ldots + \alpha_n T(\mathbf{v}_n),$$

where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$.

15.28 Let V and V' be vector spaces and let $T : V \to V'$ be a linear transformation. Prove that if W' is a subspace of V', then $T^{-1}(W')$ is a subspace of V.

15.29 Prove that there exists a bijective linear transformation $T : \mathbf{R}^2 \to \mathcal{C}$, where $\mathcal{C} = \{a + bi \ : \ a, b \in \mathbf{R}\}$ is the set of complex numbers.

15.30 For vector spaces V and V', let T_1 and T_2 be linear transformations from V to V'. Define $T_1 + T_2 : V \to V'$ as

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v}).$$

Prove that $T_1 + T_2$ is also a linear transformation.

15.31 Let $W = \left\{ \begin{bmatrix} a & b \\ 0 & a+b \end{bmatrix} \ : \ a, b \in \mathbf{R} \right\}$.

(a) Prove that W is a subspace of $M_2(\mathbf{R})$

(b) Prove that there exists a bijective linear transformation $T : \mathbf{R}^2 \to W$.

15.32 For the 2×3 matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \end{bmatrix}$, a function $T : \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $T(\mathbf{u}) = A\mathbf{u}$, where $\mathbf{u}$ is a 3×1 column vector in $\mathbf{R}^3$.

(a) Determine $T(\mathbf{u})$ for $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$.

(b) Prove that T is a linear transformation.

15.33 Let $D : \mathbf{R}[x] \to \mathbf{R}[x]$ be the differentiation linear transformation defined by

$$D(c_0 + c_1 x + \ldots + c_n x^n) = c_1 + 2c_2 x + \ldots + nc_n x^{n-1}.$$

Determine each of the following.

(a) $D(W)$, where $W = \{a + bx \ : \ a, b \in \mathbf{R}\}$.

(b) $D(W)$, where $W = \mathbf{R}$.

(c) $\ker(D)$.

15.34 Let $T : M_2(\mathbf{R}) \to M_2(\mathbf{R})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & a \\ c & c \end{bmatrix}$$

and consider the subset $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \ : \ a, d \in \mathbf{R} \right\}$ of $M_2(\mathbf{R})$.

(a) Prove that W is a subspace of $M_2(\mathbf{R})$.

(b) Determine the subspace $T(W)$ of $M_2(\mathbf{R})$.

(c) Determine the subspace $\ker(T)$ of $M_2(\mathbf{R})$.

15.35 For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect, or (3) S is false and the proof is incorrect. Explain which of these occurs.

S: Let V be a vector space. If $\mathbf{u}$ is a vector of V such that $\mathbf{u}+\mathbf{v} = \mathbf{v}$ for some $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$.

Proof. Assume that $\mathbf{u}+\mathbf{v} = \mathbf{v}$ for some $\mathbf{v} \in V$. Then we also know that $\mathbf{z}+\mathbf{v} = \mathbf{v}$, where $\mathbf{z}$ is the zero vector of V. Hence $\mathbf{u} + \mathbf{v} = \mathbf{z} + \mathbf{v}$. By Exercise 15.6, $\mathbf{u} = \mathbf{z}$ and so $\mathbf{u} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$. ■