The Invertible Matrix Theorem. Let A be an $n \times n$ matrix. Then the following are equivalent.

a. A is an invertible matrix.

b. A is row equivalent to I_n, the $n \times n$ identity matrix.

c. A has n pivot positions.

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e. The columns of A form a linearly independent set.

f. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

g. The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^n$.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

j. There is an $n \times n$ matrix C such that $CA = I_n$.

k. There is an $n \times n$ matrix D such that $AD = I_n$.

l. A^T is an invertible matrix.

Proof: The theorem is proven using this scheme:

a.⇒j. Definition of "invertible."

j.⇒d. Exercise 23, Section 2.1

d.⇒e. Box, p. 66

e.⇒f. Theorem 12, Section 1.9

f.⇒d. Theorem 11, Section 1.9

d.⇒c. Exercise 23, Section 2.2

c.⇒b. Exercise 23, Section 2.2

b.⇒a. Exercise 23, Section 2.2

a.⇒k. Definition of "invertible."

k.⇒g. Exercise 24, Section 2.2

g.⇒h. Theorem 4, Section 1.4

h.⇒i. Theorem 12, Section 1.9

i.⇒g. Definition of "onto."

g.⇒a. Exercise 24, Section 2.2

a.⇒l. Theorem 6, Section 2.2

l.⇒a. Theorem 6, Section 2.2

Theorem: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function for which $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof: Suppose that T is invertible. Let $\mathbf{b} \in \mathbb{R}^n$, and let $\mathbf{x} = S(\mathbf{b})$. Then $T(\mathbf{x}) = T(S(\mathbf{b})) = \mathbf{b}$, so $\mathbf{b}$ is in the range of T for all $\mathbf{b} \in \mathbb{R}^n$, and T is onto. Thus by IMT, A is invertible.

Suppose that A is invertible, and let $S(\mathbf{x}) = A^{-1}(\mathbf{x})$. Then S is a linear transformation, $S(T(\mathbf{x})) = A^{-1}(A\mathbf{x}) = \mathbf{x}$, and $T(S(\mathbf{x})) = A(A^{-1}\mathbf{x}) = \mathbf{x}$. So T is invertible.

Suppose that $S : \mathbb{R}^n \to \mathbb{R}^n$ and $U : \mathbb{R}^n \to \mathbb{R}^n$, and that T is invertible with $S(T(\mathbf{x})) = \mathbf{x}$ and $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. let $\mathbf{v} \in \mathbb{R}^n$. Since T is onto, there is an $\mathbf{x} \in \mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{v}$. So $S(\mathbf{v}) = \mathbf{x}$ and $U(\mathbf{v}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Thus $S = U$, and the inverse of T is unique.