Taylor and Maclaurin Series

Questions:

Given a function $f(x)$, can we represent $f(x)$ by a power series? If so, how can we find the power series?

Answer:

Start by recalling Taylor and Maclaurin polynomials.

Definition: If *f* has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the n^{th} **Taylor polynomial** for *f* at c. If c=0, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

is called the the n^{th} **Maclaurin polynomial** for *f.*

Example 1: $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

Let's find some Maclaurin polynomials!

$$f(0) = \frac{1}{1-0} = 1$$

$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$, so $f'(0) = 1$

$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$, so $f''(0) = 2$

$f'''(x) = -6(1-x)^{-4}(-1) = 6(1-x)^{-4}$, so $f'''(0) = 6$, and we see that the pattern is

$f^{(n)}(x) = -n!(1-x)^{-(n+1)}(-1) = n!(1-x)^{-(n+1)}$, so $f^{(n)}(0) = n!$

Thus the n^{th} Maclaurin polynomial for $f(x) = \frac{1}{1-x}$ is

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + x + \frac{2}{2} x^2 + \frac{6}{6} x^3 + ... + \frac{n!}{n!} x^n = 1 + x + x^2 + x^3 + ... + x^n$$

which is just the n^{th}partial sum of the power series we found for $f(x)$.

Theorem:

If f is represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for all x in an open interval containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ... =$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

NOTE: This is a big "if."

Definition:

If f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n =$$

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... + \frac{f^{(n)}(c)}{n!}(x-c)^n + ...$$

is the **Taylor series for *f* at *c***. If $c = 0$, then it is the **Maclaurin series for *f* at *c***.

Finding Taylor Series:

- **Example: The Maclaurin series for $f(x) = e^x$**

$f(0) = e^0 = 1$

$f'(x) = e^x$, so $f'(0) = 1$

$f''(x) = e^x$, so $f''(0) = 1$

$f'''(x) = e^x$, so $f'''(0) = 1$

$f^{(n)}(x) = e^x$, so $f^{(n)}(0) = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots =$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

In this case, we showed earlier that the function $f(x) = e^x$ is represented by this same series. That is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots, \; -\infty < x < \infty$$

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Series[Exp[x], {x, 0, 10}]
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$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40\,320} + \frac{x^9}{362\,880} + \frac{x^{10}}{3\,628\,800} + O[x]^{11}$$

- **Example: The Maclaurin series for $f(x) = \sin x$**

$f(0) = \sin 0 = 0$

$f'(x) = \cos x$, so $f'(0) = 1$

$f''(x) = -\sin x$, so $f''(0) = 0$

$f'''(x) = -\cos x$, so $f'''(0) = -1$

$f^{(4)}(x) = \sin x$, so $f^{(4)}(0) = 0$

and this pattern now repeats.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots =$$

$$0 + x + \frac{0\,x^2}{2!} - \frac{x^3}{3!} + \frac{0\,x^4}{4!} + \frac{x^5}{5!} + \frac{0\,x^6}{6!} - \frac{x^7}{7!} + \frac{0\,x^8}{8!} + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

In this case, we do not know whether the function $f(x) = \sin x$ is represented by a power series centered at $c = 0$. If it is, then it must be the one we have just found.

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Series[Sin[x], {x, 0, 10}]
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$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362\,880} + O[x]^{11}$$

- **Example: The Taylor series for $f(x) = \ln x$ centered at $c = 1$**

$f(1) = \ln 1 = 0$

$f'(x) = x^{-1}$, so $f'(1) = 1$

$f''(x) = -x^{-2}$, so $f''(1) = -1$

$f'''(x) = 2\,x^{-3}$, so $f'''(1) = 2$

$f^{(4)}(x) = -6\,x^{-4}$, so $f^{(4)}(1) = -6$

$f^{(5)}(x) = 24\,x^{-5}$, so $f^{(5)}(1) = 24$

$f^{(n)}(x) = (-1)^{n+1}(n-1)!\,x^{-n}$, so $f^{(n)}(1) = (-1)^{n+1}(n-1)!$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n =$$

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \frac{f^{(5)}(1)}{5!}(x-1)^5 + \ldots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \ldots =$$

$$1(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5 + \ldots + \frac{(-1)^n(n-1)!}{n!}(x-1)^n + \ldots =$$

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + \ldots + \frac{(-1)^n}{n}(x-1)^n + \ldots =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}(x-1)^n$$

In this case, we do not know whether the function $f(x) = \ln x$ is represented by a power series centered at $c = 1$. If it is, then it must be the one we have just found.

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Series[Log[x], {x, 1, 10}]
```

$$(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + \frac{1}{7}(x-1)^7 - \frac{1}{8}(x-1)^8 + \frac{1}{9}(x-1)^9 - \frac{1}{10}(x-1)^{10} + O[x-1]^{11}$$

Questions:

Given a function $f(x)$, can we represent $f(x)$ by a power series? If so, how can we find the power series?

Answer:

Recall the Taylor remainder.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1},$$ where *z* is between *x* and *c*.

Theorem:

The equality

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

holds if and only if there is a *z* between *x* and *c* such that

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} = 0$$

for all *x* in an interval about *c*.

Example: The Maclaurin series for $f(x) = \sin x$

$f(0) = \sin 0 = 0$

$f'(x) = \cos x$, so $f'(0) = 1$

$f''(x) = -\sin x$, so $f''(0) = 0$

$f'''(x) = -\cos x$, so $f'''(0) = -1$

$f^{(4)}(x) = \sin x$, so $f^{(4)}(0) = 0$

and this pattern now repeats.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-0)^{n+1} = \frac{g(z)}{(n+1)!}x^{n+1},$$

where $g(z)$ is one of the four functions $\cos z$, $\sin z$, $-\cos z$, and $-\sin z$. In any case, $|g(z)| \le 1$, and

$$\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} \left| \frac{g(z)}{(n+1)!} x^{n+1} \right| = \lim_{n\to\infty} \frac{|g(z)||x|^{n+1}}{(n+1)!} \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for all *x* since the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges for all *x*. Thus the Maclaurin series represents $\sin x$ for all *x*; that is,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots, \; -\infty < x < \infty$$

Example: The Maclaurin series for $f(x) = \cos x$

Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots, \; -\infty < x < \infty$$

differentiating immediately gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots, \; -\infty < x < \infty$$

Example: $f(x) = \sin \sqrt{x}$

Since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots, \; -\infty < x < \infty$$

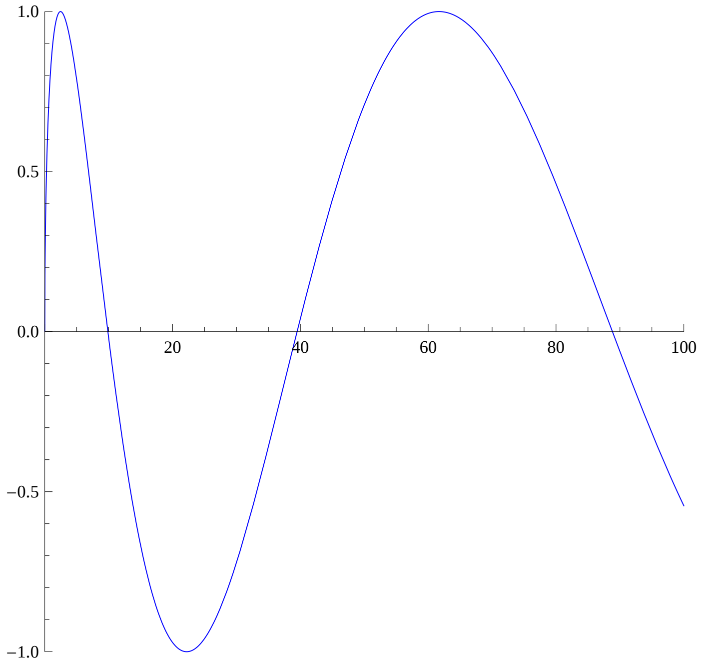
we can substitute to find that

$$\sin \sqrt{x} = \sin\left(x^{1/2}\right) = x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \frac{x^{7/2}}{7!} + \frac{x^{9/2}}{9!} + \ldots, \; 0 \le x < \infty$$

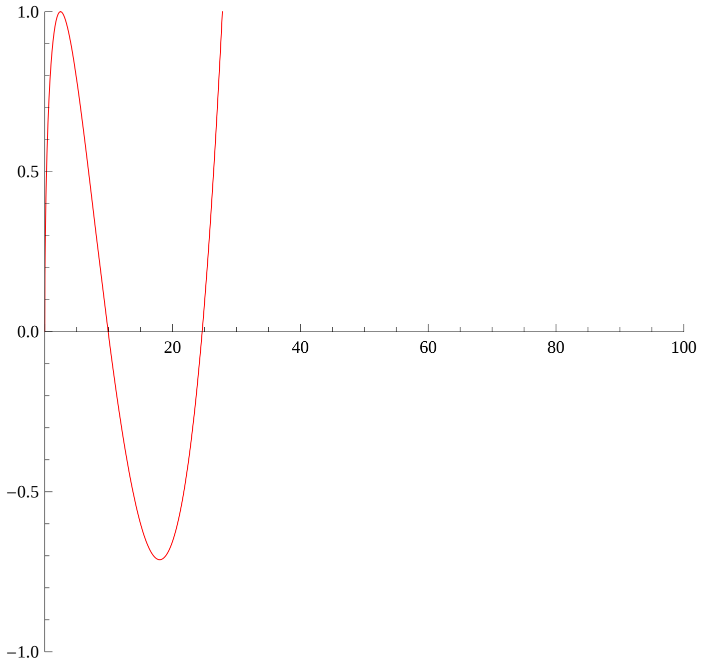
```
Series[Sin[Sqrt[x]], {x, 0, 5}]
```

$$\sqrt{x} - \frac{x^{3/2}}{6} + \frac{x^{5/2}}{120} - \frac{x^{7/2}}{5040} + \frac{x^{9/2}}{362\,880} + O[x]^{11/2}$$

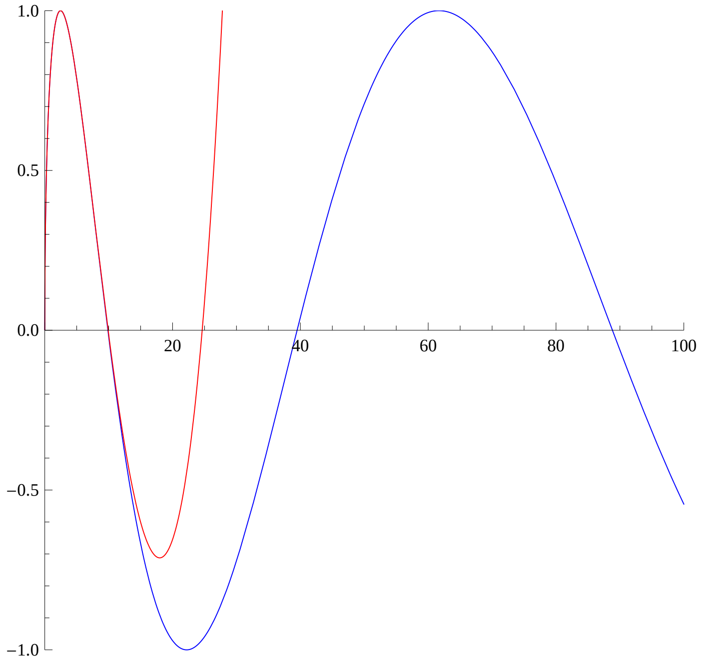
```
Plot[Sin[Sqrt[x]], {x, 0, 100}, AxesOrigin → {0, 0},
 PlotRange → {{0, 100}, {-1, 1}}, PlotStyle → {Blue}, AspectRatio → 1]
```




```
Plot[Evaluate[Normal[Series[Sin[Sqrt[x]], {x, 0, 5}]]], {x, 0, 100}, AxesOrigin → {0, 0},
 PlotRange → {{0, 100}, {-1, 1}}, PlotStyle → {Red}, AspectRatio → 1]
```



```
Show[Plot[Sin[Sqrt[x]], {x, 0, 100}, AxesOrigin → {0, 0},
  PlotRange → {{0, 100}, {-1, 1}}, PlotStyle → {Blue}, AspectRatio → 1],
 Plot[Evaluate[Normal[Series[Sin[Sqrt[x]], {x, 0, 5}]]], {x, 0, 100}, AxesOrigin → {0, 0},
  PlotRange → {{0, 100}, {-1, 1}}, PlotStyle → {Red}, AspectRatio → 1]]
```



Find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left(\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right) - x\right) =$$

$$\lim_{x\to 0} \frac{1}{x^3}\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\right) = \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \frac{x^6}{9!} + \ldots\right) = -\frac{1}{3!} = -\frac{1}{6}$$

```
Series[(Sin[x] - x) / x^3, {x, 0, 5}]
```

$$-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + O[x]^6$$

Cool Example: $f(x) = e^{ix}$

We define $i = \sqrt{-1}$, so

$$i = \sqrt{-1}$$

$$i^2 = \left(\sqrt{-1}\right)^2 = -1$$

$i^3 = i^2 \times i = -i$

$i^4 = i^2 \times i^2 = 1$

$i^5 = i^4 \times i = i$

and so on. Since

$$e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + ...$$

we have

$$e^{i x} = 1 + i x + \frac{1}{2!} (i x)^2 + \frac{1}{3!} (i x)^3 + \frac{1}{4!} (i x)^4 + \frac{1}{5!} (i x)^5 +$$
$$\frac{1}{6!} (i x)^6 .+ \frac{1}{7!} (i x)^7 .+ \frac{1}{8!} (i x)^8 .+ \frac{1}{9!} (i x)^9 .+ \frac{1}{10!} (i x)^{10} . ... =$$
$$1 + i x + \frac{1}{2!} i^2 x^2 + \frac{1}{3!} i^3 x^3 + \frac{1}{4!} i^4 x^4 + \frac{1}{5!} i^5 x^5 +$$
$$\frac{1}{6!} i^6 x^6 + \frac{1}{7!} i^7 x^7 + \frac{1}{8!} i^8 x^8 + \frac{1}{9!} i^9 x^9 + \frac{1}{10!} i^{10} x^{10} + ... =$$
$$\left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \frac{1}{8!} x^8 - \frac{1}{10!} x^{10} + ...\right) +$$
$$i \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \frac{1}{9!} x^9 - ...\right) =$$

$\cos x + i \sin x$

That is, $e^{i x} = \cos x + i \sin x$

If we let $x = \pi$ in this equation, we have

$e^{i \pi} = \cos \pi + i \sin \pi = -1$, or

$e^{i \pi} + 1 = 0$

```
Exp[ⅈ π]
```

```
-1
```

```
Exp[ⅈ x] // ExpToTrig
```

```
Cos[x] + ⅈ Sin[x]
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