MATH 450H Fall 2009

Problem Set 7

Problem 1 *Consider the initial boundary value problem:*

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2}, \quad t \geq 0, \quad 0 \leq x \leq L \\ u(0,t) &= u(L,t) = 0, \quad t \geq 0 \\ u(x,0) &= f(x), \quad 0 \leq x \leq L \end{aligned} \tag{1}$$

When solving using separation of variables, we assume $u(x,t) = v(x)\, g(t)$, *which implies*

$$\frac{1}{kg(t)}\frac{dg}{dt} = \frac{1}{v(x)}\frac{d^2 v}{dx^2} = \lambda.$$

Show that the separation constant $\lambda \in \mathbb{R}$ *can only be positive.*

Problem 2 *Show that the set of functions* $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ *is orthogonal on* $[0, L]$. *Specifically, show*

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \left\{ \begin{array}{ll} 0, & n \neq m \\ \frac{L}{2}, & n = m \end{array} \right. .$$

Problem 3 *Verify that the solution to (1) meets the given boundary conditions.*

Problem 4 *Given (1) with* $k = L = 1$ *and* $f(x) = x - x^2$, *view the solution with Mathematica. (Consider viewing various time profiles in some fashion.)*

Problem 5 *Determine if the set of functions* $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ *is orthogonal on* $[0, L]$.

Problem 6 *Change the boundary conditions in (1) to*

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(L,t) = 0$$

and solve the problem using separation of variables. Verify the boundary conditions and plot the solution.