MATH 305
Spring mass system

Consider a system of $n = 3$ masses and $m = 4$ springs. (See the accompanying figure in your class notes). Using Hooke's law and Newton's second law, we (eventually) determine that the following system models the positions of the three masses.

$$M\mathbf{x}''(t) = -B^T K B \mathbf{x}(t) \tag{1}$$

The matrix M is an $n \times n$ diagonal matrix comprised of the individual masses.

$$M = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix}$$

The matrix B is a sparse $m \times n$ matrix representing the "elongation" of each spring from equilibrium.

$$B = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{bmatrix}$$

Lastly, K is an $m \times m$ diagonal matrix containing the spring constants.

$$K = \begin{bmatrix} k_1 & & & \\ & k_2 & & \\ & & k_3 & \\ & & & k_4 \end{bmatrix}$$

Note that if the masses of matrix M are all equal to the same value $m \neq 0$, then the system defined can be rewritten as

$$\mathbf{x}''(t) = -\frac{1}{m} B^T K B \mathbf{x}(t),$$

with symmetric matrix $A = -\frac{1}{m} B^T K B$. We can use techniques learned in class lecture to solve the symmetric system

$$\mathbf{x}''(t) = A\mathbf{x}(t). \tag{2}$$

Example 1 *Suppose all masses are equal to one and the spring constants and initial positions are*

$$k_1 =, 1 \quad k_2 = 10, \quad k_3 = 2, \quad k_4 = 3$$
$$\mathbf{x}(0)^T = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \end{bmatrix}$$

Verify that the corresponding system of differential equations is

$$\mathbf{x}''(t) = \begin{bmatrix} -11 & 10 & 0 \\ 10 & -12 & 2 \\ 0 & 2 & -5 \end{bmatrix} \mathbf{x}(t).$$

Because the resulting symmetric matrix A is real, symmetric, has 3 distinct eigenvalues, we can use orthogonality to find the vector solution $\mathbf{x}(t)$. Noting that the eigenvalues are negative and the system is second order leads to the periodicity we expect. Specifically, with eigenvalues λ_i and eigenvectors v_i:

$$\begin{aligned} \mathbf{x}(t) &= \left[c_1 \cos\left(\sqrt{-\lambda_1} t\right) + c_2 \sin\left(\sqrt{-\lambda_1} t\right)\right] \mathbf{v}_1 + \\ &\quad \left[c_3 \cos\left(\sqrt{-\lambda_2} t\right) + c_4 \sin\left(\sqrt{-\lambda_2} t\right)\right] \mathbf{v}_2 + \\ &\quad \left[c_5 \cos\left(\sqrt{-\lambda_3} t\right) + c_6 \sin\left(\sqrt{-\lambda_3} t\right)\right] \mathbf{v}_3 \end{aligned}$$

Computations and results are explained in Lab 3.