# MATH 305
## Transforming periodic functions

Let $f(t)$ be a periodic function with period $P$. Then for $n \in \mathbb{Z}$, $f(t) = f(t + nP)$. We now define the window of the periodic function $f$ as

$$f_P(t) = \begin{cases} f(t), & 0 \le t < P \\ 0, & t \ge P \end{cases}.$$

Using this definition, note that we can write

$$\begin{aligned} f_P(t - kP) &= \begin{cases} f(t - kP), & 0 \le t - kP < P \\ 0, & t - kP \ge P \end{cases} \\ &= \begin{cases} f(t - kP), & kP \le t < (k+1)P \\ 0, & t \ge (k+1)P \end{cases}, \end{aligned}$$

for all $k \in \mathbb{N}$. Then $f$ can be written in the form

$$f(t) = \sum_{k=0}^{\infty} f_P(t - kP). \tag{1}$$

Using the Heaviside function defined in class lecture, we can write (1) in yet another form, this one suitable for transformation.

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} H(t - kP) f_P(t - kP) \Rightarrow \\ L[f(t)] &= \sum_{k=0}^{\infty} e^{-(kP)s} L[f_P(t)] \\ &= \sum_{k=0}^{\infty} e^{-(kP)s} \int_0^{\infty} e^{-st} f_P(t)\, dt \\ &= \sum_{k=0}^{\infty} e^{-(kP)s} \int_0^{P} e^{-st} f(t)\, dt \\ &= \int_0^{P} e^{-st} f(t)\, dt \sum_{k=0}^{\infty} e^{-(Ps)k} \qquad (2) \end{aligned}$$

Note that the derivation of (1) and (2) above may take some time to verify. After doing so, note that the infinite sum in (2) is geometric and converges when $e^{-sP} < 1$

to

$$\sum_{k=0}^{\infty} e^{-(Ps)k} = \frac{1}{1-e^{-Ps}}.$$

Then the periodic function $f$ with period $P$ has the following Laplace transform.

$$L\left[f\left(t\right)\right] = \frac{L\left[f_P\left(t\right)\right]}{1-e^{-Ps}} = \frac{\int_0^P e^{-st} f\left(t\right) dt}{1-e^{-Ps}}$$