MATH 305
Mixing problem

Consider a closed system of three brine tanks, with volumes V_1, V_2 and V_3. The inflow to tank 1 is the outflow from tank 3; the inflow to tank 2 is the outflow from tank 1; and the inflow to tank three is the outflow from tank 2. With all three flow rates equal to r, the resulting system of differential equations model the weights of the salt in each tank at time t.

$$\begin{aligned} w_1' &= -\frac{r}{V_1}w_1 + \frac{r}{V_3}w_3 \\ w_2' &= \frac{r}{V_1}w_1 - \frac{r}{V_2}w_2 \\ w_3' &= \frac{r}{V_2}w_2 - \frac{r}{V_3}w_3 \end{aligned}$$

With $V_1 = V_3 = 50$ gallons, $V_2 = 25$ gallons, and $r = 10$ gallons per minute, the associated system is

$$\mathbf{w}'(t) = \begin{bmatrix} -.2 & 0 & .2 \\ .2 & -.4 & 0 \\ 0 & .4 & -.2 \end{bmatrix} \mathbf{w}(t).$$

Given the following eigenpairs of the matrix, determine the real solution of the system $\mathbf{w}(t)$ and its limiting value (as $t \to \infty$).

$$\begin{aligned} \lambda_1 &= 0, \quad \mathbf{v}_1^T = \begin{bmatrix} 2, & 1, & 2 \end{bmatrix} \\ \lambda_2 &= -.4 + .2i, \quad \mathbf{v}_2^T = \begin{bmatrix} -1-i, & -1+i, & 2 \end{bmatrix} \end{aligned}$$

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Solution

Note that for λ_2, $p = -.4$ and $q = .2$. For the associated eigenvector, we see that

$$\begin{aligned}\begin{bmatrix} -1-i \\ -1+i \\ 2 \end{bmatrix} &= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + i\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \mathbf{a} + i\mathbf{b}.\end{aligned}$$

Then, the real solutions corresponding to the complex eigenpairs are

$$\mathbf{w}_2(t) = e^{-.4t}\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\cos(.2t) - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\sin(.2t)\right)$$

and

$$\mathbf{w}_3(t) = e^{-.4t}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\cos(.2t) + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\sin(.2t)\right).$$

Superpositioning these with the solution corresponding to λ_1 and $\mathbf{v}_1$ gives the full solution

$$\mathbf{w}(t) = c_1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_2\mathbf{w}_2(t) + c_3\mathbf{w}_3(t).$$

This can be rewritten as

$$\begin{aligned}\mathbf{w}(t) &= \begin{bmatrix} 2c_1 + e^{-.4t}\left[(-c_2 - c_3)\cos(.2t) + (c_2 - c_3)\sin(.2t)\right] \\ c_1 + e^{-.4t}\left[(-c_2 + c_3)\cos(.2t) + (-c_2 - c_3)\sin(.2t)\right] \\ 2c_1 + e^{-.4t}\left[2c_2\cos(.2t) + 2c_3\sin(.2t)\right] \end{bmatrix} \\ &= \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}.\end{aligned}$$

Note that $(w_1 + w_2 + w_3)(t) = 2c_1 + c_1 + 2c_1 = 5c_1$; i.e. the total weight of solute in the tanks at any given time is constant, as expected.

Also note that the steady state (limiting) distribution of solute in the tank is

$$\lim_{t\to\infty} \mathbf{w}(t) = \begin{bmatrix} 2c_1 \\ c_1 \\ 2c_1 \end{bmatrix},$$

and is independent of the initial distribution. In other words, eventually the solute in the tanks will become evenly mixed throughout the three tanks; $\frac{2}{5}$ in tanks one and two (each), and $\frac{1}{5}$ in tank 2.