Characterizing The Impulse Function

Definition 1 $d_{c,\varepsilon}(t) := \begin{cases} \frac{1}{\varepsilon}, & c - \frac{\varepsilon}{2} \leq t \leq c + \frac{\varepsilon}{2} \\ 0, & else \end{cases}$

Theorem 2 *If f is continuous on $[a, b]$, with $a < b$, then $\exists\ c \in [a, b]$ such that*

$$\int_a^b f(t)\, dt = f(c)(b - a).$$

This is known as the mean value theorem (MVT) for integrals.

Proof. Omitted. ■

Now, consider the continuous function $\phi(t)$ on the interval $\left[c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right]$ such that $\phi(t) \to 0$ as $t \to \infty$. As with any continuous function, the MVT ensures the existence of some $t^* \in \left[c - \frac{\varepsilon}{2}, c + \frac{\varepsilon}{2}\right]$ such that

$$\int_{c-\varepsilon/2}^{c+\varepsilon/2} \phi(t)\, dt = \phi(t^*)\left[\left(c + \tfrac{\varepsilon}{2}\right) - \left(c - \tfrac{\varepsilon}{2}\right)\right] = \varepsilon\phi(t^*).$$

Then

$$\phi(t^*) = \frac{1}{\varepsilon}\int_{c-\varepsilon/2}^{c+\varepsilon/2} \phi(t)\, dt = \int_{c-\varepsilon/2}^{c+\varepsilon/2} \frac{1}{\varepsilon}\phi(t)\, dt = \int_0^\infty d_{c,\varepsilon}(t)\,\phi(t)\, dt.$$

This, in turn, implies that

$$\lim_{\varepsilon\to 0}\phi(t^*) = \lim_{\varepsilon\to 0}\int_0^\infty d_{c,\varepsilon}(t)\,\phi(t)\, dt.$$

Noting that $t^* \to c$ as $\varepsilon \to 0$ we see that

$$\lim_{\varepsilon\to 0}\phi(t^*) = \phi(c) = \int_0^\infty \lim_{\varepsilon\to 0} d_{c,\varepsilon}(t)\,\phi(t)\, dt.$$

Having already characterized the "function" $\delta(t - c)$ as

$$\int_0^\infty \delta(t - c)\,\phi(t)\, dt = \phi(c),$$

we have that

$$\int_0^\infty \delta(t - c)\,\phi(t)\, dt = \int_0^\infty \lim_{\varepsilon\to 0} d_{c,\varepsilon}(t)\,\phi(t)\, dt.$$

This somewhat justifies the following popular characterization of $\delta(t - c)$,

$$\delta(t - c) = \lim_{\varepsilon\to 0} d_{c,\varepsilon}(t) = \begin{cases} \infty, & t = c \\ 0, & else \end{cases},$$

as well as the name "impulse function".