MATH 305
Heat flow

Consider one-dimensional heat flow in a thin rod with no loss of heat through the rod's lateral surface. If we divide the bar into $n = 6$ "cells" (for example) of equal length, each has an interface at each end – either between cells or at the end of the rod. This construction leads to $m = 7$ interfaces, in this example. We assume the temperatures at the ends of the rod are zero. Using temperature differences across cells, Fourier's law of heat conduction, and the conservation of heat energy, we (eventually) determine that the following system models the heat flow.

$$C\mathbf{u}'(t) = -B^T K B \mathbf{u}(t) \tag{1}$$

The matrix C is an $n \times n$ diagonal matrix comprised of "heat flow proportionality constants" determined by the material comprising each cell.

$$C = \begin{bmatrix} c_1 & & & & & \\ & c_2 & & & & \\ & & c_3 & & & \\ & & & c_4 & & \\ & & & & c_5 & \\ & & & & & c_6 \end{bmatrix}$$

The matrix B is a sparse $m \times n$ matrix representing heat flow from one cell to another (or the ends of the rod).

$$B = \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & -1 & 1 & \\ & & & & -1 & 1 \\ & & & & & -1 \end{bmatrix}$$

Lastly, K is an $m \times m$ diagonal matrix containing the "thermal conductivities" of each face.

$$K = \begin{bmatrix} k_1 & & & & & & \\ & k_2 & & & & & \\ & & k_3 & & & & \\ & & & k_4 & & & \\ & & & & k_5 & & \\ & & & & & k_6 & \\ & & & & & & k_7 \end{bmatrix}$$

Now, if we let $c := c_1 = \cdots = c_6$ and define $A = -\frac{1}{c}B^TKB$, we note that the resulting system is a symmetric, 6 dimensional, and 1st order.

$$\begin{aligned} A &= -\frac{1}{c}B^TKB \Rightarrow \\ A^T &= \left(-\frac{1}{c}B^TKB\right)^T \\ &= -\frac{1}{c}B^TK^TB \\ &= -\frac{1}{c}B^TKB \\ &= A \end{aligned}$$

Then we can use techniques learned in class lecture to solve the symmetric system

$$\mathbf{u}'(t) = A\mathbf{u}(t). \tag{2}$$

Example 1 *Suppose all flow proportionality constants are equal to one and the conductivities and initial temperatures are*

$$k_1 = k_2 = k_4 = k_6 = 1, \quad k_3 = k_5 = 4, \quad k_7 = 3$$
$$\mathbf{u}(0)^T = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 & 0 \end{bmatrix}$$

Verify that the corresponding system of differential equations is

$$\mathbf{u}'(t) = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -5 & 4 & & & \\ & 4 & -5 & 1 & & \\ & & 1 & -5 & 4 & \\ & & & 4 & -5 & 1 \\ & & & & 1 & -4 \end{bmatrix} \mathbf{u}(t).$$

Because the resulting symmetric matrix A is real, symmetric, has 6 distinct eigenvalues, we can use orthogonality to find the vector solution $\mathbf{u}(t)$. Computations and results are explained in Lab 3.