MATH 201
The Fundamental Theorem of Calculus
Solutions

The Fundamental Theorem of Calculus, discovered by Isaac Newton and Gottfried Leibniz in the 17th century has been called one of the greatest discoveries in the history of science.

$$\int_a^b f(x)\,dx = \int_a^b \frac{dF}{dx}dx = F(x)]_a^b = F(b) - F(a)$$

Problem 1 *The area of an oil spill is increasing at a rate of* $25t\ ft^2/s$*,* t *seconds after the spill. Find the increase in the area of the spill between times* $t = 2$ *and* $t = 4$*.*

Solution:

$$\begin{aligned} A(4) - A(2) &= \int_2^4 \frac{dA}{dt}dt \\ &= \int_2^4 25t dt \\ &= \left[\frac{25t^2}{2}\right]_2^4 \\ &= \frac{25}{2}\left[t^2\right]_2^4 = \frac{25}{2}(16-4) = 150ft^2 \end{aligned}$$

Problem 2 *A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30p and 5:30p, the rate* $R(t)$ *at which cars enter the highway is given by the formula*

$$R(t) = 100\left(1 - .0001t^2\right)$$

cars per minute, where t *is the time (in minutes) since 4:30p. Estimate the number of cars that enter the highway during the rush hour.*

Solution:

$$\begin{aligned} &\int_0^{60} \left(100 - .01t^2\right) dt \\ =& \left[100t - \frac{.01t^3}{3}\right]_0^{60} \\ =& \ 100(60) - \frac{.01(60)^3}{3} \\ =& \ 5280 \text{ cars.} \end{aligned}$$

Problem 3 *Let* $f(x) = \int_1^x (2t+3)\,dt$. *Use the FTOC given above to find another expression for* $f(x)$. *Then compute* $f'(x)$.

Solution:

$$\begin{aligned} f(x) &= \left[t^2+3t\right]_1^x \\ &= \left(x^2+3x\right) - \left(1^2+3\right) \\ &= x^2+3x-4 \Rightarrow \\ f'(x) &= 2x+3 \end{aligned}$$

Problem 4 *Let* $f(x) = \int_1^{3x^2} (2t+3)\,dt$. *Use the FTOC given above to find another expression for* $f(x)$. *Then compute* $f'(x)$.

Solution:

$$\begin{aligned} f(x) &= \left[t^2+3t\right]_1^{3x^2} \\ &= \left[\left(3x^2\right)^2 + 3\left(3x^2\right)\right] - \left(1^2+3\right) \\ &= 9x^4+9x^2-4 \Rightarrow \\ f'(x) &= 36x^3+18x \\ &= \left[2\left(3x^2\right)+3\right](6x) \end{aligned}$$

With $u(x) = 3x^2$,

$$f'(x) = (2[u(x)]+3)\frac{du}{dx}$$

Problem 5 *Compute* $\frac{d}{dx}\int_0^{2x} \sin\left(t^2\right)dt$.

Solution:

$$\begin{aligned} \frac{d}{dx}\int_0^{2x} \sin\left(t^2\right)dt &= \sin\left[(2x)^2\right]\frac{d}{dx}[2x] \\ &= \sin\left(4x^2\right)(2) \\ &= 2\sin\left(4x^2\right). \end{aligned}$$