MATH 105
Final exam review solutions

Problem 1 *(Section 1.5: Limits.) Substituting* 1 *for* x *in the expression leads to* $\frac{0}{0}$*, which is indefinite (more work is needed):*

$$\begin{aligned}\lim_{x\to 1}\frac{(\sqrt{x}-1)}{(x-1)}\cdot\frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} &= \\ \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x}+1)} &= \\ \lim_{x\to 1}\frac{1}{\sqrt{x}+1} &= \frac{1}{2}.\end{aligned}$$

Problem 2 *(Section 1.5: Infinite Limits.) To solve this, we note the highest powers of* x *in both the numerator and denominator. Noting that* $3>1$*, we realize this is an infinite limit and thus the function has no horizontal asymptotes. We resort to a table or graph (omitted here) to determine*

$$\lim_{x\to+\infty}\frac{-x^3+2x+1}{x-3}=-\infty.$$

Problem 3 *(Section 1.6: One-Sided Limits and Continuity.) Recall that a function* $f(x)$ *is continuous at* $x=c$ *if*

$$\lim_{x\to c}f(x)=f(c).$$

Therefore, to determine if the function

$$f(x)=\begin{cases}2x^2-x, & x<3\\ 3-x, & x\geq 3\end{cases}$$

is continuous at $x=3$ *for example, we compute the appropriate limits:*

$$\begin{aligned}\lim_{x\to 3^-}\left(2x^2-x\right) &= 2(3)^2-(3)=15,\\ \lim_{x\to 3+}(3-x) &= 3-3=0,\end{aligned}$$

Because the one-sided limits are not equal, $\lim_{x\to 3}f(x)$ *DNE. Then this two sided limit cannot equal* $f(3)$*. Our conclusion is that the function is not continuous at* $x=3$*.*

Problem 4 *(Section 2.1: The Derivative.) Recall that*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h.}$$

Then, for $f(x) = x^2$,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h) = 2x. \end{aligned}$$

Problem 5 *(Section 2.2: Techniques of Differentiation.)*

$$\begin{aligned} v(t) &= s'(t) = 3t^2 - 12t + 9 \Rightarrow \\ s'(4) &= 3(4)^2 - 12(4) + 9 = 9. \end{aligned}$$

Problem 6 *(Section 2.3: Product and Quotient Rules; Higher-Order Derivatives.)*

$$\begin{aligned} y'(x) &= x^2(3) + (2x)(3x+1) \\ &= 3x^2 + 6x^2 + 2x \\ &= 9x^2 + 2x \Rightarrow \\ y''(x) &= 18x + 2. \end{aligned}$$

Problem 7 *(Section 2.4: The Chain Rule.)*

$$\begin{aligned} f'(x) &= \frac{(x-1)^2(3) - (3x-2)[2(x-1)(1)]}{(x-1)^4} \\ &= \frac{3(x-1) - 2(3x-2)}{(x-1)^3} \\ &= \frac{3x - 3 - 6x + 4}{(x-1)^3} \\ &= \frac{1-3x}{(x-1)^3} \Rightarrow \\ f''(x) &= \frac{(x-1)^3(-3) - (1-3x)\left[3(x-1)^2(1)\right]}{(x-1)^6} \end{aligned}$$

Problem 8 *(Section 2.6: Implicit Differentiation.)*

$$\begin{aligned} x^2\frac{dy}{dx} + 2xy + 2y\frac{dy}{dx} &= 3x^2 \Rightarrow \\ \frac{dy}{dx}\left(x^2+2y\right) &= 3x^2 - 2xy \Rightarrow \\ \frac{dy}{dx} &= \frac{3x^2-2xy}{x^2+2y} \end{aligned}$$

Problem 9 *(Section 3.1: Increasing and Decreasing Functions; Relative Extrema.)*
We first find critical values:

$$\begin{aligned} f(t) &= \left(3-2t-t^2\right)^{1/2} \Rightarrow \\ f'(t) &= \frac{1}{2}\left(3-2t-t^2\right)^{-1/2}(-2-2t) \\ &= \frac{-2-2t}{2\sqrt{3-2t-t^2}}. \end{aligned}$$

We find points in the domain such that $f'(t) = 0$ and $f'(t)$ does not exist.

$$\begin{aligned} f'(t) &= 0 \Rightarrow \\ -2-2t &= 0 \Rightarrow \\ t &= -1. \end{aligned}$$

Also,

$$\begin{aligned} f'(t)\ \ \mathit{DNE} &\Rightarrow \\ 2\sqrt{3-2t-t^2} &= 0 \Rightarrow \\ 3-2t-t^2 &= 0 \Rightarrow \\ (1-t)(3+t) &= 0 \Rightarrow \\ t &= 1, -3. \end{aligned}$$

Our critical values are $t = -3, -1, 1$. Testing around these values with the first derivative may indicate relative extrema. Noting that the domain of $f(t)$ is $-3 \leq t \leq 1$ limits our testing possibilites. Setting up a first derivative test about the value $t = -1$, we note that

$$f'(-2) > 0 \text{ and } f'(0) < 0.$$

Therefore, $t = -1$ is a relative maximum. Moreover, the interval of increase is $(-3, -1)$ and the interval of decrease is $(-1, 1)$.

Problem 10 *(Section 3.2: Concavity and Points of Inflection.) We first look for the locations of possible points of inflection.*

$$\begin{aligned} f'(x) &= 12x^5 - 20x^3 + 7 \Rightarrow \\ f''(x) &= 60x^4 - 60x^2 = 0 \Rightarrow \\ x^2(x^2-1) &= 0 \Rightarrow x = 0, -1, 1. \end{aligned}$$

Testing around these possibilities with the second derivative gives that

$$f''(-2) > 0, \quad f''\left(-\frac{1}{2}\right) < 0, \quad f''\left(\frac{1}{2}\right) < 0, \quad f''(2) > 0.$$

Then points of inflection are loctated at $x = -1, 1$ only. Moreover, the function is concave up on $(-\infty, -1)$ and $(1, \infty)$. It is concave down on $(-1, 0)$ and $(0, 1)$.

Problem 11 *(Section 3.4: Optimization.) To answer this question, we find the critical values of the function on the given interval.*

$$\begin{aligned} S'(t) &= 3t^2 - 21t + 30 = 0 \Rightarrow \\ 3(t-5)(t-2) &= 0 \Rightarrow t = 2, 5. \end{aligned}$$

We then test these critical values, along with the interval endpoints $t = 1, 6$, to determine the largest and smallest:

$$S(1) = 40.5, \quad S(2) = 46, \quad S(5) = 32.5, \quad S(6) = 38.$$

Then the slowest traffic occurs at 5 p.m. $(t = 5)$, while the fastest occurs at 2 p.m. $(t = 2)$.

Problem 12 *(Section 4.3: Differentiation of Logarithmic and Exponential Functions.)*

$$\begin{aligned} f(x) &= \ln\left(2x^3 + 1\right) \Rightarrow \\ \frac{df}{dx} &= \frac{6x^2}{2x^3+1}. \end{aligned}$$

Also,

$$\begin{aligned} f(x) &= e^{x^2+1} \Rightarrow \\ f'(x) &= e^{x^2+1}(2x). \end{aligned}$$

Problem 13 *(Section 5.1: Antidifferentiation: The Indefinite Integral.)*

$$\begin{aligned} & \int \left(\frac{x^3 + 2x - 7}{x} \right) dx \\ = & \int \left(\frac{x^3}{x} + \frac{2x}{x} - \frac{7}{x} \right) dx \\ = & \int \left(x^2 - 2 - \frac{7}{x} \right) dx \\ = & \frac{x^3}{3} - 2x - 7 \ln |x| + C. \end{aligned}$$

Problem 14 *(Section 5.2: Integration by Substitution.) Choosing $u = \ln x$, we note that $\frac{du}{dx} = \frac{1}{x}$, and "$dx = xdu$." Then*

$$\int \frac{(\ln x)^2}{x} dx = \int \frac{u^2}{x} xdu = \int u^2 du = \frac{u^3}{3} + C = \frac{1}{3} (\ln x)^3 + C.$$

Problem 15 *(Section 5.3: The Definite Integral and the Fundamental Theorem of Calculus.) We determine the net change in mass of the protein during the first two hours by computing the following definite integral.*

$$\int_0^2 \frac{-30}{(t+3)^2} dt$$

Choosing $u = t + 3$, we note that $\frac{du}{dt} = 1$, and "$dt = du$." Then

$$m(t) = \int \frac{-30}{u^2} du = \int -30u^{-2} du = \frac{-30u^{-1}}{-1} = \frac{30}{u} = \frac{30}{t+3}.$$

Evaluating between the integration limits gives the answer:

$$m(2) - m(0) = \frac{30}{2+3} - \frac{30}{0+3} = 6 - 10 = -4.$$

This means that the mass has decreased by 4 grams during this time frame.

Problem 16 *(Section 5.4. Applying Definite Integration: Area Between Curves and Average Value.) We first find where the curves intersect:*

$$\begin{aligned} x^3 &= x^2 \Rightarrow \\ x^3 - x^2 &= 0 \Rightarrow \\ x^2 (x-1) &= 0 \Rightarrow x = 0, 1. \end{aligned}$$

These are our limits of integration. Noting that $x^2 > x^3$ on $(0,1)$, we have that

$$\begin{aligned} \textit{Area} = & \int_0^1 \left(x^2 - x^3\right) dx \\ = & \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 \\ = & \left(\frac{1^3}{3} - \frac{1^4}{4}\right) - \left(\frac{0^3}{3} - \frac{0^4}{4}\right) \\ = & \frac{1}{12}. \end{aligned}$$

Problem 17 *(Appendix A3. Evaluating Limits with L'Hôpital's Rule). Substituting* 1 *for* x *in the expression leads to* $\frac{0}{0}$*, which is indefinite (more work is needed).*

$$\begin{aligned} & \lim_{x\to 1} \frac{x^5 - 3x^4 + 5x - 3}{4x^5 + 2x^3 - 5x^2 - 1} \\ = & \lim_{x\to 1} \frac{5x^4 - 12x^3 + 5}{20x^4 + 6x^2 - 10x} \\ = & \frac{5(1)^4 - 12(1)^3 + 5}{20(1)^4 + 6(1)^2 - 10(1)} = -\frac{1}{8}. \end{aligned}$$