Isomorphism and a Few Example Applications of Graphs

Isomorphism

The prefix iso means same, and morph means form. Isomorphic graphs are graphs that have the same form. Being able to show that two graphs have the same form means that you can apply things you have learned about one graph to the other.

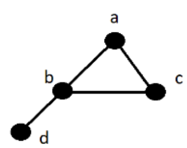
Isomorphic – graph G1 and graph G2 are isomorphic if there is a mapping of the vertices in G1 to the vertices in G2 such that the vertex and edge sets are identical.

To show that two graphs are isomorphic, we just need to find the mapping described in the definition. To show that they are not isomorphic, we have to explain how we know that such a mapping cannot exist.

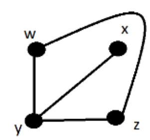
Example 1

Determine whether or not the graphs pictured below are isomorphic

a) Graph 1



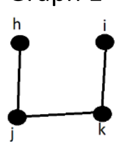
Graph 2



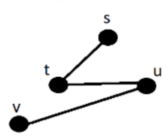
Answer: They have the same form – they're both a triangle *(abc and wyz)* with an edge off one of the triangle's vertices *(bd and yx)*

More formally, G1=(V1,E1), V1={a,b,c,d}, E1={{a,b},{a,c},{b,c},{b,d}}
G2=(V2,E2), V2 = {w,x,y,z}, E2 = {{w,y},{w,z},{y,x},{y,z}}
These graphs are isomorphic, with a mapping a->z, b->y, c->w, d->x
V1 = {a,b,c,d}, E1={{a,b},{a,c},{b,c}, {b,d} }
V2 = {w,x,y,z}, E2 = {{w,y},{w,z},{y,x},{y,z}}

b) Graph 1



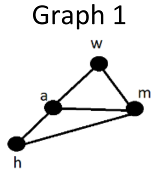
Graph 2



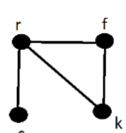
Answer: They have the same form – a line of four vertices

More formally, G1 = (V1,E1), V1={h,i,j,k},E1={{h,j},{j,k},{k,i}}
G2=(V2,E2), V2={s,t,u,v},E2={{s,t},{t,u},{u,v}}
These graphs are isomorphic, with a mapping h->s,i->v,j->t,k->u
V1={h,i,j,k}, E1={{h,j} {j,k},{k,i}}
V2={s,t,u,v}, E2={{s,t},{t,u},{u,v}}

c) Graph 1

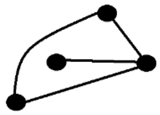


Graph 2

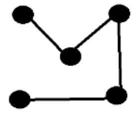


Answer: These graphs aren't isomorphic. Graph 1 has more edges than Graph 2.

d) Graph 1

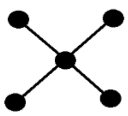


Graph 2

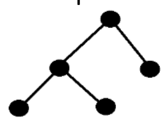


Answer: These graphs aren't isomorphic. The graph on the left has fewer vertices than the one on the right.

e) Graph 1



Graph 2



Answer: These graphs aren't isomorphic. The graph on the left has a vertex with degree 4. None of the vertices on the right have degree 4.

When examining graphs that have the same form, a natural question is: how many different (non-isomorphic) graphs are there that meet certain criteria?

Let's start small, and begin with simple (no loops or multi-edges) undirected graphs.

Before we start drawing graphs, it would help to know what the maximum number of edges a simple undirected graph can have. Recall that each edge in an undirected graph is a set of two vertices. So counting the number of possible edges in a graph with n vertices is equivalent to counting the number of ways we can create a set/group of 2 elements a set of n elements. So the maximum number of edges in a simple undirected graph with n vertices is ${}_nC_2$.

So, for example, a simple undirected graph with 5 vertices can have at most ${}_5C_2 = 10$ edges.

Ok.. Back to counting graphs…

How many non-isomorphic undirected simple graphs are there with 0 vertices?

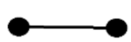
Answer: 1. I've drawn it below.

How many non-isomorphic undirected simple graphs are there with 1 vertex?

Answer: 1. I've drawn it below.

How many non-isomorphic undirected simple graphs are there with 2 vertices?

Answer: 2. I've drawn them below.



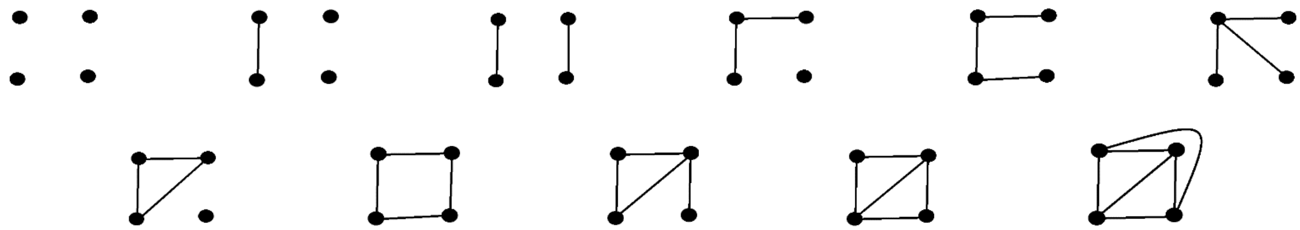
See if you can draw all of the non-isomorphic undirected simple graphs with 3 vertices. *(There are 4)*

And see if you can draw all of the non-isomorphic undirected simple graphs with 4 vertices.

After you've drawn them, check them against my drawings on the next page.

Below are the four undirected simple graphs are there with 3 vertices.

And here are the 11 undirected simple graphs are there with 4 vertices.



The number of different graphs with 0, 1, 2, 3, 4.... vertices is a sequence 1, 1, 2, 4, 11… but the pattern isn't obvious (the next value in the sequence – the number of non-isomorphic graphs with 5 vertices – is 34). If you're interested in reading more about this sequence and other sequences, check out one of the oldest sites on the internet, the Online Encyclopedia of Integer Sequences – here's the entry for this particular sequence.

A few graph applications / classic graph problems

Graphs are made up of vertices and edges. So graphs can be applied to problems where there are things (vertices) and relationships between pairs of things (edges).

A classic problem posed in many introductory graph theory texts is the handshaking problem, a version of which is given below. This problem comes from back in the old days when people could safely come within six feet of one another and have slight physical contact (a handshake or fist bump) without perpetuating the spread of an active pandemic.

Problem 1: Handshaking problem

Professor CS hosts a gathering of prospective computer science students, current majors, and current faculty. At the gathering, people meet and introduce themselves to one another with a handshake and a short conversation. As people leave, Professor CS's assistant asks each person how many people they shook hands with, and records their answers. Without looking at the list of numbers, Professor CS states that there are an even number of odd numbers in the list (e.g. if the list of hands shook was 6, 8, 10, 9, 4, 5, 2, 7, Professor CS would be wrong because that list contains 3 odd numbers). Is Professor CS guaranteed to be correct?

Answer: This problem involves objects = people = vertices, and relationships among pairs of objects = handshakes = edges. So the problem can be viewed as a graph. Note that the graph that represents this problem contains no loops (people don't shake hands with themselves, so there are no edges from a vertex to itself). Asking each person for the number of people they shook hands with is equivalent to obtaining degree of each vertex in the graph. Each edge in a graph contributes +1 to the degree of each of the two vertices that the edge links. So, if we were to add up the degrees of all of the vertices, the sum would equal twice the number of edges in the graph.

$$\sum \deg(v_i) = 2 \cdot |E|$$

Thus the sum of the degrees is an even number.

In order for the sum of a list of values to be even, the list must contain an even number of odd values. So Professor CS is correct!

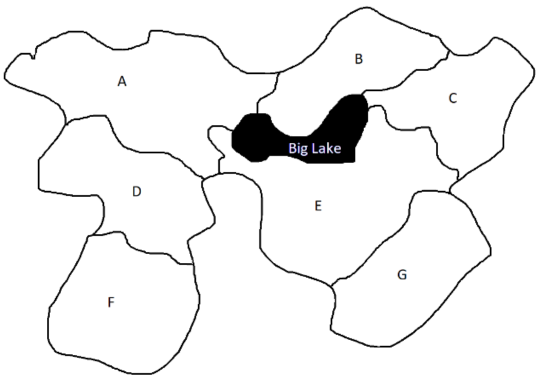
Key takeaway from this problem: the formula above

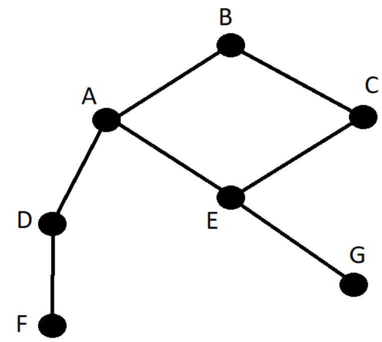
Problem 2: Graph Coloring

Q is creating a virtual continent full of imaginary countries that border one another. Q wants to create a map of the continent and color each country on the map so that no countries that share a border have the same color. What criteria must hold in order to be able to color the map using only two colors?

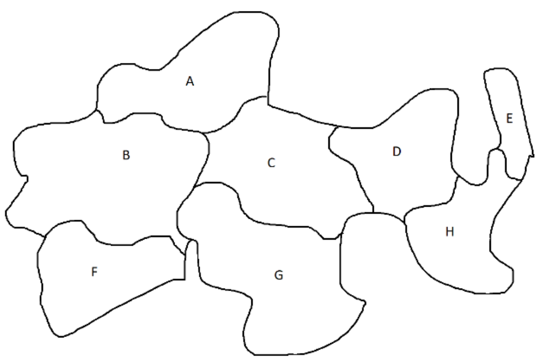
This problem involves objects = countries = vertices, and relationships between those objects = borders between two countries = edges. So we can view this problem as a graph.

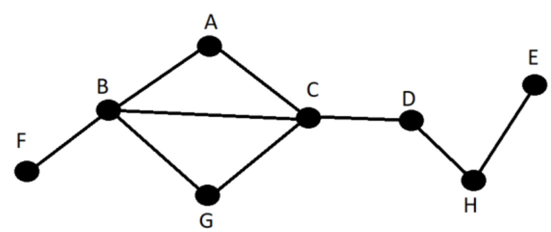
For example, if Q had the continent below, the corresponding graph is to its right.





Or if Q had the continent below, the corresponding graph is given here.

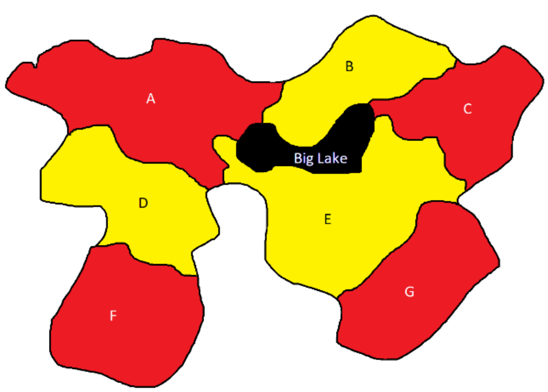


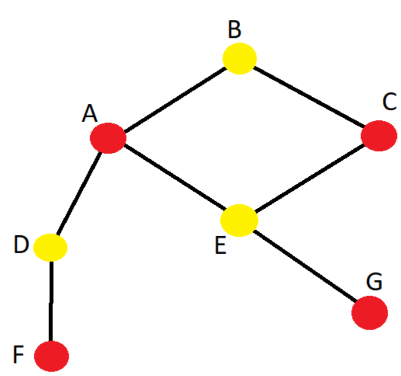


This problem is a specific example of a more general class of problems called coloring problems where you're asked to find the minimum number of colors that can be used to assign each vertex a color such that no two adjacent vertices are assigned the same color.

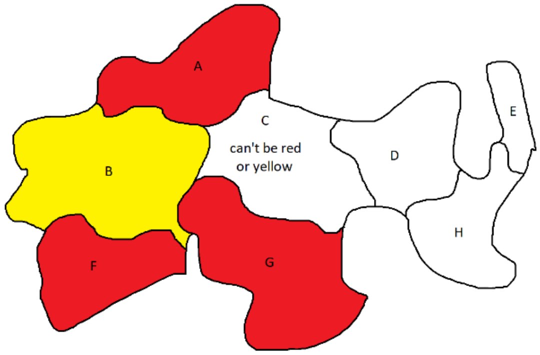
The term for a two-colorable graph is **bipartite** (two parts). A graph is bipartite if and only if there are no odd length circuits in the graph.

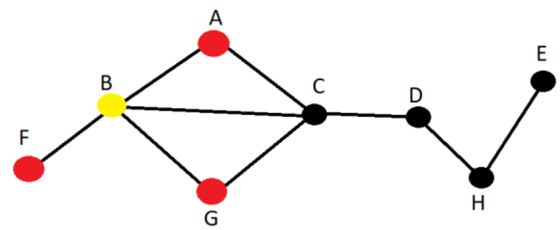
The first example graph above has no circuits of odd length.





The second example graph above has odd length circuits, so it is NOT bipartite. We would need more than two colors to color the vertices so that no adjacent vertices are the same color.

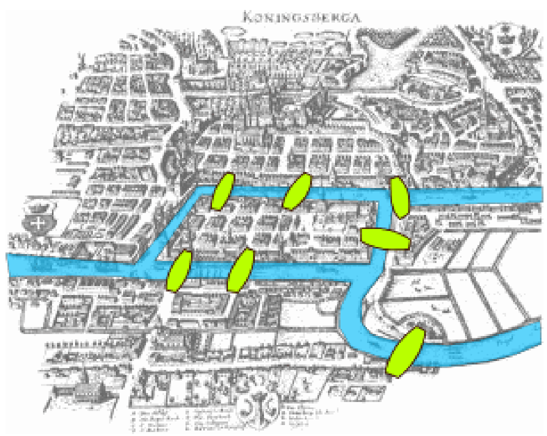




Key takeaway from the problem: a graph is bipartite if and only if there are no odd length circuits in the graph.

Problem 3: The Königsberg Bridge Problem

Königsberg was a city in Prussia (it's now the Russian city of Kaliningrad). In 1736, Leonhard Euler solved the following problem based on the map of Königsberg and its 7 bridges shown below.

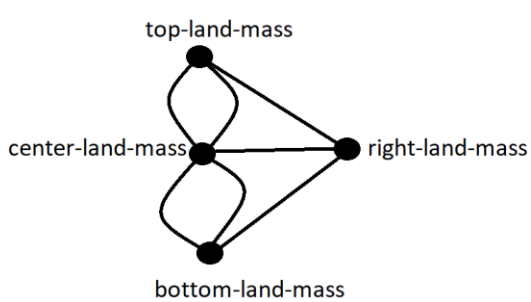


By Bogdan Giuşcă - Public domain (PD),based on the image, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=112920

The posed problem was: Can one take a walking tour of Königsberg such that each of the 7 bridges is crossed exactly once?

Again, this problem is made up of objects = land masses = vertices, and relationships between pairs of objects = bridges between land masses = edges. Notice that this graph is a multigraph – there are land masses linked by more than one edge.

Here is picture of a graph of the map from above.



The Königsberg bridge problem, in terms of the graph is: is there a path that traverses each edge exactly once. Due to Leonhard Euler's initial solution to this problem, a path through a graph that traverses each edge exactly once is called an **Euler path**.

Euler's key observations were these:

If the path begins at an even degree vertex, it must end there.

For any even degree vertex that the path that is not the starting, the path cannot end at that vertex.

If the path begins at an odd degree vertex, it cannot end there.

For any odd degree vertex that is not the starting vertex, the path MUST end at that vertex.

From the first two observations, a graph with all even degree vertices will have an Euler path. The Euler path will begin and end at the same vertex.

From the last two observations, a graph containing odd degree vertices will only have an Euler path if it has exactly 2 odd degree vertices – one for the path to start from, and one for the path to end with.

In the Königsberg bridge graph, there are 4 vertices with odd degree, so there is no path that traverses each edge exactly once.

Key takeaway: A graph will contain an Euler path if and only if there are 0 or 2 vertices of odd degree.

Due to the definition in Problem 3 of an Euler path, you might be thinking, what do we call paths that visit each vertex exactly once?

A path through the graph that visits each vertex exactly once is called a **Hamiltonian path**, named after William Rowan Hamilton (not Alexander or the musical). Unlike Euler paths, there's not a quick way to determine if a graph has one.

As with each discrete mathematics topic discussed in this course, we just had time to scratch the surface in our introduction to graph theory. It's yet another field in which mathematicians devote their entire careers. Hopefully this introduction piqued your interest and you'll explore avenues for further study of these topics in the future. For the time being, however, let's concentrate of finishing this course...

It's now time to complete your last graded homework assignment, and study for test 4!