Don't be a Square

Team # 24086

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Abstract

In order to create a brownie pan that both heats evenly and minimizes wasted space in an oven, we begin by modeling the arrangement of pans in an oven. We limit ourselves to even sided, regular polygon shaped pans that are arranged parallel to the back of the oven. This assumption allows us to restrict the arrangement of pans to two formations: skew and grid. We define conditions for each of the even sided n-gons ($4 < n < 12$) in which the skew pattern produces a more optimal arrangement than the grid pattern. With these conditions in place, we create a formula for the number of pans that can fit in an oven with a given pan area, oven length, and oven width.

With a formula for calculating the maximum number of pans that can fit in the oven in place, we create an agent-based model in which the oven, air, brownies, and pan are represented as cubic cells in a virtual environment. The behavior of each cell is governed by the first law of thermodynamics (i.e. the energy level of each cell approaches equilibrium). We describe the equations governing each cell and provide three-dimensional simulations of brownies cooking using NetLogo to plot the average temperature around the edges vs. the average temperature of the middle.

With these models in place, we are able to determine the best pan to use in a given $L \times W$ oven. We find that a hexagon pan provides the greatest balance of spatial optimization and heat distribution.

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1 Introduction

Originally used as a light confection for boxed lunches, the brownie has been a favorite of American dessert connoisseurs since its creation in the late 19th century [1]. A pan of brownies is cooked according to the first law of thermodynamics. The first law of thermodynamics states that the increase in internal energy of a body is equal to the heat supplied to the body minus work done by the body. This means that the oven, pan, brownie system will approach equilibrium as components with higher temperatures transfer heat to the components with cooler temperatures.

From this observation, we can divide brownies into two distinct categories - edge pieces and interior pieces. An edge piece will absorb heat directly from the sides of the pan while an interior piece will be heated by surrounding pieces through diffusion. A special case of the edge category is the corner. At a corner, the brownie is heated by two adjacent pan edges which can result in over-cooking.

A circle pan can be thought to have only one edge and no corners. While this pan does ensure that the brownie is cooked evenly, this shape is not efficient with respect to space utilization in a square oven.

Thus we wish to design a pan that:

- Maximizes even distribution of heat (H) for the pan
- Maximizes the number of pans that can fit in the oven (N)

2 Outline of Approach

1. We begin by constructing a three dimensional, agent-based model for heat transfer.
2. Using NetLogo and the agent-based model we created a virtual oven.
3. By using our virtual oven, we can determine the temperature of the brownies in pans of varying shapes.
4. In order to determine the number of pans that can fit into the oven, we explore stacking patterns of even sided, regular polygons and create conditions in which a skew pattern is more optimal than a grid pattern.

3 Assumptions

- Each pan in the oven contains the same volume of brownie mix.
  - This allows us to assume that if two pans are placed on the same rack, side-by-side, these brownies should bake at the same rate. Therefore, when we place the pans closer together we know that it is not the volume of brownie mix that is causing a difference in baking rate.

- The shape of the pan is either a circle or a regular polygon with number of sides $n = 4, 6, 8, 10, 12$.
  - If the pan is a circle the brownies will not be overcooked at the edges. Regular polygons have the same angle at each "corner" and therefore their browning rate at those spots will be the same and only slightly different along the edges of the pan. We chose an even number of sides for that regular polygons that each side is parallel to exactly one other side. Which allows for our next assumption.
- Pans are arranged in the oven with two sides parallel to L.
  - This, along with the previous assumption, allows us to consider only two arrangements of pans: skew and grid.
- Uniform distribution of all ingredients throughout brownie mix.
  - This allows us to assume the brownies are done when the temperature reaches 160°F.
- Brownies height is constant.
  - This assumption allows assures uniform baking of the brownie in the pan.

4 Model

4.1 Geometric Model, Grid vs. Skew

The number of pans that can fit in an $L \times W$ oven is a function of edge length and arrangement. Since we assume a pan must have two edges parallel to the back of the oven, the pans must be in a grid Figure 1 or skew Figure 2 arrangement in order to be optimal. By exploiting even sided polygons' mirror symmetry, we are able to see patterns form for when the skew pattern yields a higher number of pans that can fit in the oven. **In this section we describe conditions in which the skew pattern produces more allowable pans than the grid pattern and create a function for the maximum number of pans that can fit in a given oven**. Note for $n = 4$ edges the pans can only be arranged in a grid formation.

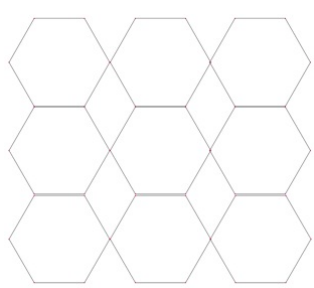

Figure 1: Grid Formation

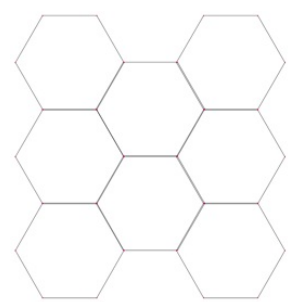

Figure 2: Skew Formation

4.1.1 Formulas and Notation Used

a	the perpendicular distance from a polygon's circumcenter to its base, also known as the apothem
c	the straight line distance between the polygon's circumcenter and a vertex, also known as the circumradius $c = \frac{s}{2 \cdot Sin\left[\frac{360}{n}\right]} = \frac{a}{Cos\left[\frac{360}{n}\right]}$
n	number of sides of the polygon
d	horizontal length of the polygon
h	vertical length of the polygon
s	the length of one of the polygon's sides
A	area of the polygon $A = \frac{1}{4}ns^2Cot\left[\frac{360}{n}\right]$

4.1.2 Conditions

Hexagon $(n = 6)$

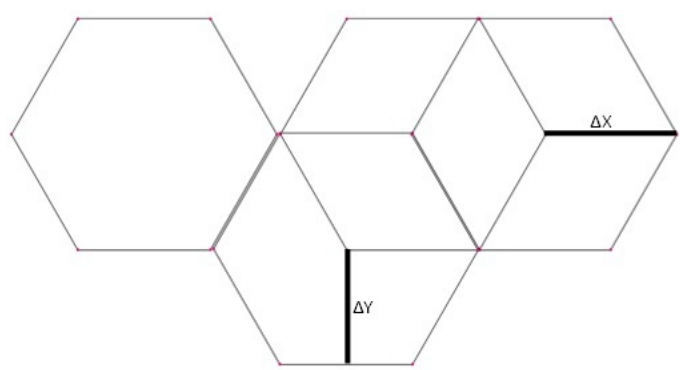

Figure 3: Hexagon

In Figure 3 above, we can label the distance added in the lateral direction by changing from grid to skew pattern as Δx, similarly the distance added in the vertical direction as Δy. We now derive the added distance in both the x and y directions.

As seen in the figure, the change in x direction is equal to the hexagon's circumradius. We note that in the case of a regular hexagon, the circumradius is equal to the hexagon's side. Thus we can say

$$\Delta x \;=\; c \;=\; s.$$

The change in the y direction is equal to the hexagon's apothem. Using the pythagorean theorem we arrive at:

$$\Delta y \;=\; a \;=\; s \cdot \frac{\sqrt{s}}{2}.$$

We summarize our results as the following conditions:

- if the horizontal length and height of the pan divides the length and height of the oven respectively, the grid pattern is optimal.
- else if there is Δx length units between the last column of pans and the oven wall and if there is δy length units between the last row of pans and the oven wall then the skew pattern is optimal.
- else the grid pattern is optimal.

Mathematically, these conditions can be expressed as:

$$\Delta x < \frac{L}{d} < d \text{ or } \Delta y < \frac{w}{h} < h \text{ where } d = 2c \text{ and } h = 2a.$$

Octagon Conditions ($n = 8$)

We see in Figure 4 that due to the octagon's shape, when arranged in a skew pattern the arrangement maintains none of its original grid pattern. This means that both conditions for Δx and Δy must be met in order for the skew pattern to yield more allowable pans.

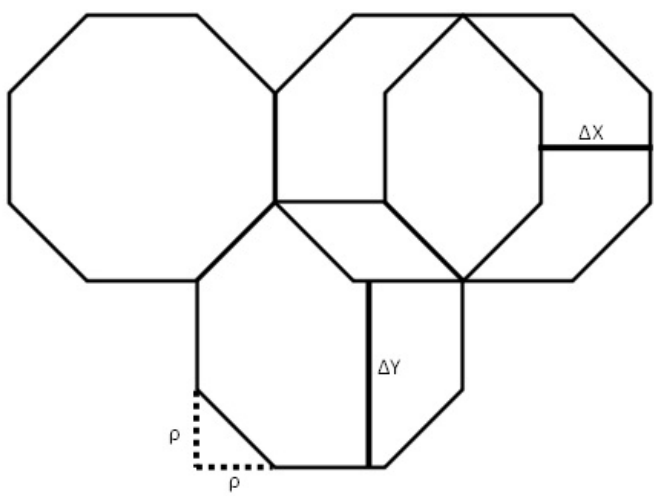

Figure 4: Octagon

We observe that the change in the x direction is equal to the length of the octagon's side

$$\Delta x \;=\; s.$$

To find the change in the y direction we must first find δy. Using the pythagorean theorem we observe that $\delta y = \frac{s}{\sqrt{2}}$. From here, we are able to define Δy as:

$$\Delta y = s + \delta y = s + \frac{\sqrt{s}}{2}.$$

Thus, we arrive at the following conditions:

$$\Delta x <= \frac{L}{d} \text{ and } \Delta y <= \frac{W}{h} \text{ where } d = h = 2a.$$

Decagon Conditions ($n = 10$)

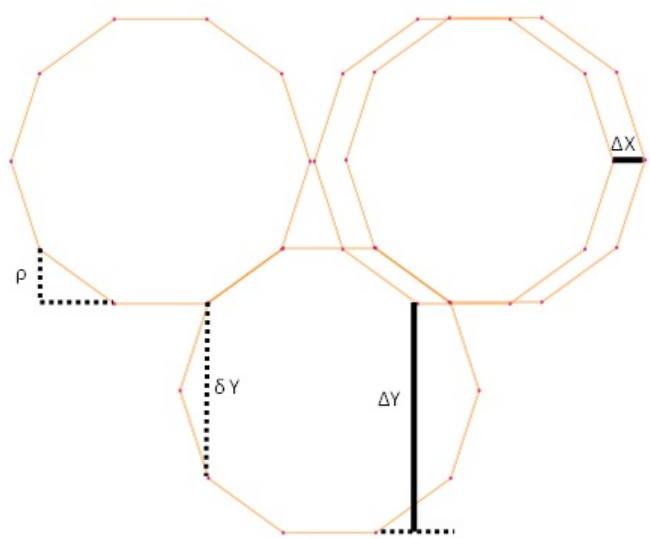

Figure 5: Decagon

In order to create conditions for the decagon we must make two intermediate calculations: δy and ρ. See Figure 5. We calculate these terms by using the law of sines. We then have:

$$\begin{aligned} \delta y &= \frac{s \cdot Sin[144°]}{Sin[18°]} \\ \rho &= s \cdot Sin[36°]. \end{aligned}$$

We are then able to define Δx and Δy as follows:

$$\Delta x = 2s \cdot \sqrt{s^2 - (\delta y)^2}$$

$$\Delta y = \delta y + 2\rho.$$

Like the octagon, the decagon must meet both conditions in order for the skew pattern to produce more allowable pans than the grid pattern. We can then observe the conditions in terms of s:

$$\Delta x < \frac{L}{d} < d \text{ and } \Delta y < \frac{W}{h} < h \text{ where } d = 2ah = 2c.$$

Dodecagon Conditions ($n = 12$)

The dodecagon behaves similarly to the hexagon, in that the first row is the same in both the grid and skew arrangements. Because of this, the dodecagon can fall under the same conditions as the hexagon with different Δx and Δy values.

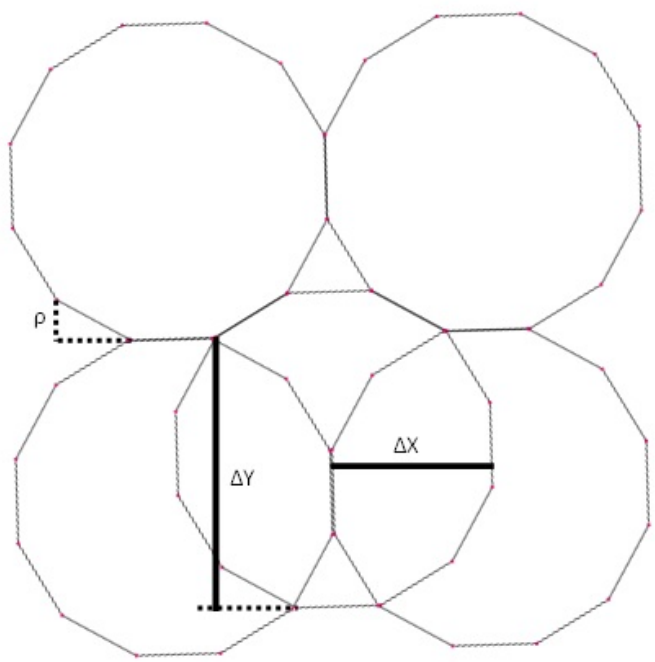

Figure 6: Dodecagon

We find Δx by making the observation in Figure 6 that Δx is equal to the apothem of the dodecagon. Thus

$$\Delta x \;=\; a \;=\; \frac{s \cdot Cot[30°]}{2}.$$

In order to calculate Δy we must make an intermediate calculation of ρ. Using the law of sines we find that:

$$\rho = s \cdot Sin[30°].$$

Thus,

$$\Delta y \;=\; 2a \;-\; \rho \;=\; \frac{2s \cdot Cot[30°]}{2} - 2s \cdot Sin[30°].$$

We can then define the following conditions for the number of dodecagon pans that can fit in an $L \times W$ pan by:

$$\Delta x < \frac{L}{d} < d \text{ and } \Delta y < \frac{W}{h} < h \text{ where } d = h = 2a.$$

Circle Conditions $(n \to \infty)$

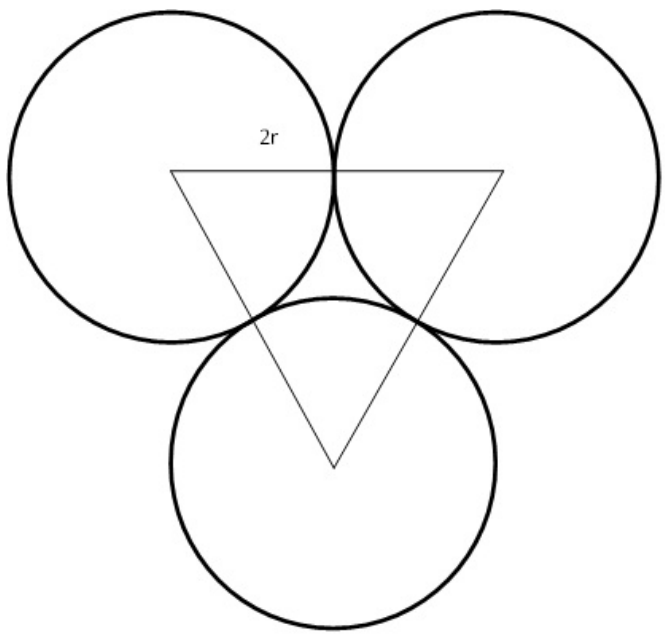

Figure 7: Circle

As we can see in figure 7, the radius of circles form an equilateral triangle. We use this observation and the pythagorean theorem to determine Δx and Δy.

$$\Delta x = r$$

$$\Delta y = r \cdot \sqrt{5}.$$

We then define the conditions for skew pattern yielding more allowable circular pans than the grid pattern by:

$$\Delta x < \frac{L}{d} < d \text{ and } \Delta y < \frac{W}{h} < h \text{ where } d = h = 2c.$$

4.1.3 Performance of Skew Arrangement

From above, we have conditions for which a polygon added in the middle of four adjacent polygons arranged in a skew formation would yield more allowable pans than the grid pattern.

Based on these findings, we create a function for the number of pans allowed in an $L \times W$ oven by:

$$F_i = \begin{cases} \lfloor \frac{L}{d} \rfloor \cdot \lfloor \frac{W}{h} \rfloor, & \text{if grid} \\ \\ \frac{5 \cdot \lfloor \frac{L}{d} \rfloor \cdot \lfloor \frac{W}{h} \rfloor}{4}, & \text{if skew.} \end{cases}$$

4.2 Agent-Based Model: Heat Flow

4.2.1 Heat Flow Equations

In order to model heat distribution through the pan, we create an angent-based model in which we represent the oven, pan, and brownies in a three-dimensional space divided into cubic cells. Each cell, with coordinates (i, j, k) has six faces, each adjacent to one of six surrounding cells $(i+1, j, k), (i-1, j, k), (i, j+1, k), (i, j-1, k), (i, j, k+1)$, and $(i, j, k-1)$. We identify every cell as being an air, pan, brown, or oven cell, with its own temperature which affects surrounding cells. Take, for example, two adjacent cells (i, j, k) and $(i, j, k+1)$. The heat flow $Q_{(i,j,k)}$ between the two cells depends on the thermal conductivity of the cells $K_{(i,j,k)}$ and $K_{(i,j,k+1)}$, as well as their current temperatures, $T_{(i,j,k)}$ and $T_{(i,j,k+1)}$.

If we consider only the heat flow from (i, j, k) to $(i, j, k+1)$, the flow is given by,

$$Q_{(i,j,k)(i,j,k+1)} = K_{(i,j,k)} \cdot (T_{(i,j,k)} - T_{(i,j,k+1)}).$$

Then, if $T_(i, j, k) > T(i, j, k+1)$ then the heat flow is positive (since the thermal conductivity constant, $K > 0$). Then (i, j, k) is losing energy and transferring that energy to $(i, j, k+1)$. To consider the total energy lost or gained by (i, j, k), we sum the heat flow from the cell to its adjacent cells with current temperatures T^t. Then, to attain the new temperature at time, $T^{t+1}_{(i,j,k)}$ of the cell,

$$T^{t+1}_{(i,j,k)} = T^t_{(i,j,k)} - \Sigma Q_{(i,j,k)}.$$

Expanding the series and factoring out $K_{(i,j,k)}$, our equation becomes,

$$\begin{aligned} T^{t+1}_{(i,j,k)} &= T^t_{(i,j,k)} - K_{(i,j,k)}(T^t_{(i,j,k)} - T_{(i+1,j,k)} + T^t_{(i,j,k)} - T_{(i-1,j,k)} + T^t_{(i,j,k)} - T_{(i,j+1,k)} + \\ &\quad T^t_{(i,j,k)} - T_{(i,j-1,k)} + T^t_{(i,j,k)} - T_{(i,j,k+1)} + T^t_{(i,j,k)} - T_{(i,j,k-1)}). \end{aligned}$$

Which simplifies to,

$$\begin{aligned} T^{t+1}_{(i,j,k)} &= T^t_{(i,j,k)} - 6K_{(i,j,k)}T^t_{(i,j,k)} + K_{(i,j,k)}(T_{(i+1,j,k)} + T_{(i-1,j,k)} + T_{(i,j+1,k)} + T_{(i,j-1,k)} + \\ &\quad T_{(i,j,k+1)} + T_{(i,j,k-1)}) \end{aligned}$$

Then, letting $S = \{T_{(i+1,j,k)}, T_{(i-1,j,k)}, T_{(i,j+1,k)}, T_{(i,j-1,k)}, T_{(i,j,k+1)}, T_{(i,j,k-1)}\}$ be the set of current temperatures for the six cells adjacent to (i, j, k),

$$T^{t+1}_{(i,j,k)} = T^t_{(i,j,k)} - 6K_{(i,j,k)}T^t_{(i,j,k)} + K_{(i,j,k)} \sum_S T.$$

We simplify to attain,

$$= K_{(i,j,k)} \sum_S T + (1 - 6K_{(i,j,k)})T^t_{(i,j,k)}.$$

We now have one equation for each cell to calculate the new temperature after heat flow.

4.2.2 NetLogo Simulation

We use an agent-based modeling software, NetLogo3D. The cells of the three-dimensional virtual environment are considered agents, called patches in the NetLogo programming language. This allows each cell to store its temperature and thermal conductivity. The adjacent cells are also identified as neighbors.

5 Results

5.1 Geometric Results

In order to test our geometric model, we tested ovens of team members to come up with the following test cases for our model.

In our tests, we did not expect to have the grid pattern as the optimal arrangement in every case. In order to test all cases we chose L and W so that the skew pattern would be most favorable.

$26\frac{1}{2} \times 25\frac{11}{16}$		
n	N	Pattern
1	4	Grid
4	4	Grid
6	4	Grid
8	4	Grid
10	4	Grid
12	4	Grid

$29\frac{1}{2} \times 25\frac{11}{16}$		
n	N	Pattern
1	4	Grid
4	6	Grid
6	6	Grid
8	4	Grid
10	4	Grid
12	4	Grid

$25\frac{1}{2} \times 24$		
n	N	Pattern
1	4	Grid
4	4	Grid
6	4	Grid
8	4	Grid
10	4	Grid
12	4	Grid

$28\frac{1}{2} \times 24$		
n	N	Pattern
1	6	Grid
4	4	Grid
6	4	Grid
8	4	Grid
10	4	Grid
12	4	Grid

In our tests, we did not expect to have the grid pattern as the optimal arrangement in every case. In order to test all cases we chose L and W so that the skew pattern would be most favorable.

66.0102×102.753		
n	N	Pattern
1	60	Grid
4	77	Grid
6	75	Skew
8	75	Skew
10	75	Skew
12	60	Grid

5.2 Agent-Based Model Results

In order to simulate our model, we used an agent-based modeling software, NetLogo3D. The cells of the three-dimensional virtual environment are considered agents, called patches in the NetLogo programming language. This allows each cell to store its temperature and thermal conductivity. The adjacent cells are also identified as neighbors.

Shape	Max Temperature (F°)	Average Temperature of Pan (F°)
Circle	243.06	220.21
Square	261.93	220.71
Hexagon	264.55	232.94
Octagon	256.98	229.92
Decagon	263.08	231.91
Dodecagon	257.12	230.99

6 Discussion

The geometric model predicts that the square out performs the other figures in household oven dimensions. The only polygon that rivals the square is the hexagon in the $29\frac{1}{2} \times 25\frac{11}{16}$ pan. When simulating the hexagon in the agent-based model, we found that the hexagon had the highest temperature around its corner. We believe these results to be skewed. In the agent-based model, a cubic cell corresponding to the pan would be affected by the oven only on its outer sides; however, in NetLogo a pan cube can be affected by the oven on multiple sides due to its jagged shape (see figure 9). To combat this issue, we considered the average temperature around the corner and

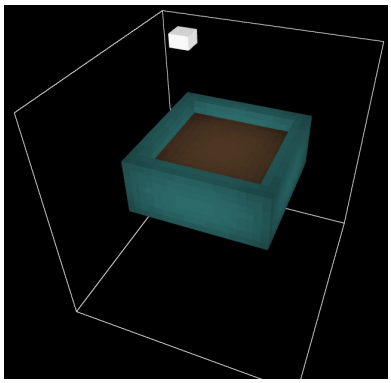

Figure 8: Square Pan

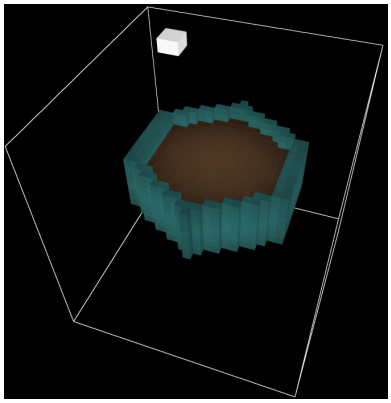

Figure 9: Hexagon Pan

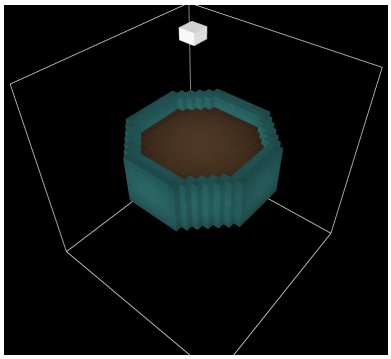

Figure 10: Octagon Pan

middle. While we believe this would have resolved the issue, the NetLogo data corresponding to these values was lost and cannot be provided. Based on observed cooking patterns, we assume corner temperatures is inversely proportional to the measure of the polygon's interior angle and for this reason believe the hexagon to be the best shape to both cook evenly and provide for the maximum number of pans.

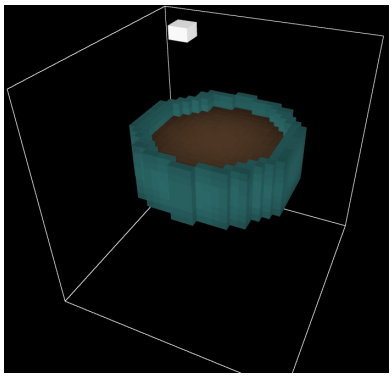

Figure 11: Decagon Pan

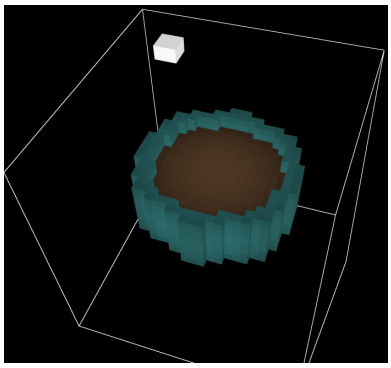

Figure 12: Dodecagon Pan

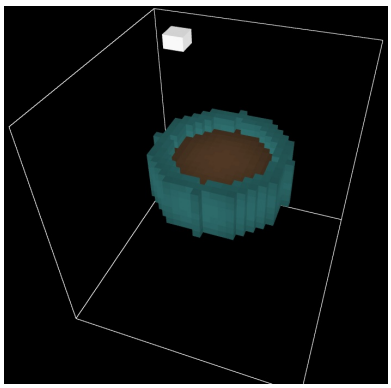

Figure 13: Circle Pan

6.1 Strengths and Weaknesses of the Model

The greatest strength of the models presented is the fact they are fairly robust to changes in oven length and oven width parameters. This is advantageous if the model were used in real-life since oven dimensions are not uniform.

Another strength is that the agent-based model can describe heat flow through any shape of

pan. Since each cell in the agent-based model is governed by the first law of thermodynamics and nothing else, this portion of the model could be used if assumptions made in the geometric model.

The greatest weakness of the two models is that the geometric model requires a specification of area, oven length, and oven width. This means that we are unable to determine a general case in which one shape out performs the other.

Another weakness of the presented models is the computation time required for the agent-based model. In order to accurately simulate the cooking process, we need to separate the virtual environment into a large number of cubic units. For more accuracy, the number of cubic units must become sufficiently large resulting in a computationally intensive method of determining the heat of the corner and the middle sections of the pan.

References

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Tired of brownies with over baked corners?

Then stop being a square and realize the power of TheHexagon™.

Our patented design ensures for a more even cooking of your favorite brownies all while ensuring that your oven rack remains open for other tasty treats.

(We recommend more brownies.)

Finally, science and cooking have come together to bring you the ultimate in brownie cooking technology. TheHexagon™ makes a great gift for friends and loved ones. So give your friends the gift of perfect brownies with TheHexagon™.

Perfection ...

in every bite.