Blammo the Human Cannonball

Blammo the Human Cannonball will be fired from a cannon and hopes to land in a small net at the opposite end of the circus arena. Your job as Blammo's manager is to do the mathematical calculations that will allow Blammo to perform his death-defying act safely. The methods that you will use are from the field of ***ballistics*** *(the study of projectile motion).*

The Problem

Blammo's cannon has a ***muzzle velocity*** of 35 m/s, which means that Blammo will leave the muzzle with that velocity. The muzzle opening will be 5 m above the ground, and Blammo's objective is to land in a net that is also 5 m above the ground and that extends a distance of 10 m between 90 m and 100 m from the cannon opening (Figure 1). Your mathematical problem is to determine the ***elevation angle*** α of the cannon (the angle from the horizontal to the cannon barrel) that will make Blammo land in the net.

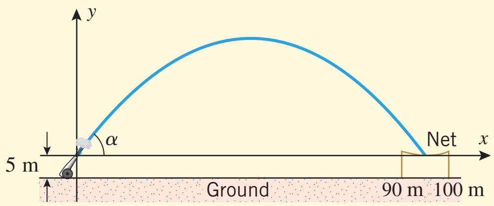

Figure 1

Modeling Assumptions

Blammo's trajectory will be determined by his initial velocity, the elevation angle of the cannon, and the forces that act on him after he leaves the muzzle. We will assume that the only force acting on Blammo after he leaves the muzzle is the downward force of the Earth's gravity. In particular, we will ignore the effect of air resistance. It will be convenient to introduce the xy-coordinate system shown in Figure 1 and to assume that Blammo is at the origin at time $t = 0$. We will also assume that Blammo's motion can be decomposed into two independent components, a horizontal component parallel to the x-axis and a vertical component parallel to the y-axis. We will analyze the horizontal and vertical components of Blammo's motion separately, and then we will combine the information to obtain a complete picture of his trajectory.

Blammo's Equations of Motion

We will denote the position and velocity functions for Blammo's horizontal component of motion by $x(t)$ and $v_x(t)$, and we will denote the position and velocity functions for his vertical component of motion by $y(t)$ and $v_y(t)$.

Since the only force acting on Blammo after he leaves the muzzle is the downward force of the Earth's gravity, there are no horizontal forces to alter his initial horizontal velocity $v_x(0)$. Thus, Blammo will have a constant velocity of $v_x(0)$ in the x-direction; this implies that

$$x(t) = v_x(0)t \tag{1}$$

In the y-direction Blammo is acted on only by the downward force of the Earth's gravity. Thus, his motion in this direction is governed by the free-fall model; hence, from (15) in Section 5.7* his vertical position function is

$$y(t) = y(0) + v_y(0)t - \tfrac{1}{2}gt^2$$

Taking $g = 9.8\ \mathrm{m/s^2}$, and using the fact that $y(0) = 0$, this equation can be written as

$$y(t) = v_y(0)t - 4.9t^2 \tag{2}$$

Exercise 1 At time $t = 0$ Blammo's velocity is 35 m/s, and this velocity is directed at an angle α with the horizontal. It is a fact of physics that the initial velocity components $v_x(0)$ and $v_y(0)$ can be obtained geometrically from the muzzle velocity and the angle of elevation using the triangle shown in Figure 2. We will justify this later in the text, but for now use this fact to show that Equations (1) and (2) can be expressed as

$$x(t) = (35\cos\alpha)t$$
$$y(t) = (35\sin\alpha)t - 4.9t^2$$

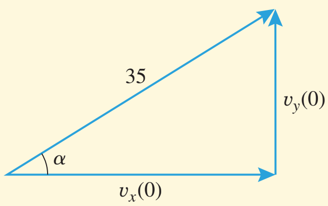

Figure 2

Exercise 2

(a) Use the result in Exercise 1 to find the velocity functions $v_x(t)$ and $v_y(t)$ in terms of the elevation angle α.

(b) Find the time t at which Blammo is at his maximum height above the x-axis, and show that this maximum height (in meters) is

$$y_{\max} = 62.5\sin^2\alpha$$

Exercise 3 The equations obtained in Exercise 1 can be viewed as parametric equations for Blammo's trajectory. Show, by eliminating the parameter t, that if $0 < \alpha < \pi/2$, then Blammo's trajectory is given by the equation

$$y = (\tan\alpha)x - \frac{0.004}{\cos^2\alpha}x^2$$

Explain why Blammo's trajectory is a parabola.

Finding the Elevation Angle

Define Blammo's ***horizontal range*** R to be the horizontal distance he travels until he returns to the height of the muzzle opening ($y = 0$). Your objective is to find elevation angles that will make the horizontal range fall between 90 m and 100 m, thereby ensuring that Blammo lands in the net (Figure 3).

*This reference to Calculus, Early Transcendentals, 9th edition, corresponds to Formula (15) in Section 4.7 of Calculus, 9th edition.

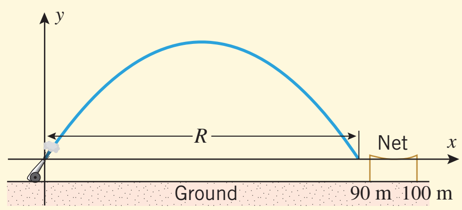

Figure 3

Exercise 4 Use a graphing utility and either the parametric equations obtained in Exercise 1 or the single equation obtained in Exercise 3 to generate Blammo's trajectories, taking elevation angles at increments of 10° from 15° to 85°. In each case, determine visually whether Blammo lands in the net.

Exercise 5 Find the time required for Blammo to return to his starting height ($y = 0$), and use that result to show that Blammo's range R is given by the formula

$$R = 125 \sin 2\alpha$$

Exercise 6

(a) Use the result in Exercise 5 to find two elevation angles that will allow Blammo to hit the midpoint of the net 95 m away.

(b) The tent is 55 m high. Explain why the larger elevation angle cannot be used.

Exercise 7 How much can the smaller elevation angle in Exercise 6 vary and still have Blammo hit the net between 90 m and 100 m?

Blammo's Shark Trick

Blammo is to be fired from 5 m above ground level with a muzzle velocity of 35 m/s over a flaming wall that is 20 m high and past a 5-m-high shark pool (Figure 4). To make the feat impressive, the pool will be made as long as possible. Your job as Blammo's manager is to determine the length of the pool, how far to place the cannon from the wall, and what elevation angle to use to ensure that Blammo clears the pool.

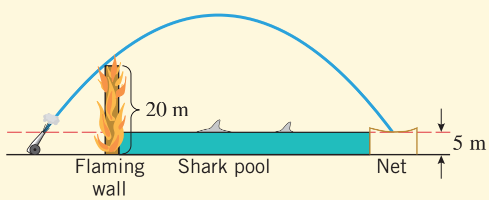

Figure 4

Exercise 8 Prepare a written presentation of the problem and your solution of it that is at an appropriate level for an engineer, physicist, or mathematician to read. Your presentation should contain the following elements: an explanation of all notation, a list and description of all formulas that will be used, a diagram that shows the orientation of any coordinate systems that will be used, a description of any assumptions you make to solve the problem, graphs that you think will enhance the presentation, and a clear step-by-step explanation of your solution.

Module by: *John Rickert, Rose-Hulman Institute of Technology*
Howard Anton, Drexel University