Math 300

Section 6.1 Inner Product, Length, and Orthogonality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then the inner product (or dot product) of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} \cdot \mathbf{v}$, is

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Properties of the Inner Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$

c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The length (or norm) of a vector $\mathbf{v}$ in $\mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

A unit vector has length 1. To find a unit vector in the direction of $\mathbf{v}$ is to normalize the vector $\mathbf{v}$. To do this, we multiply $\mathbf{v}$ by the reciprocal of its length.

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then the distance between $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u} - \mathbf{v}\|$.

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Pythagorean Theorem Generalized

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

If W is a subspace of $\mathbb{R}^n$, then the vector $\mathbf{z}$ in $\mathbb{R}^n$ is orthogonal to W if $\mathbf{z} \cdot \mathbf{w} = 0$ for all vectors $\mathbf{w}$ in W. The set of all vectors $\mathbf{z}$ in $\mathbb{R}^n$ that are orthogonal to W is called $W^\perp$, the orthogonal complement of W.

Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix. Then

$$(Row\ A)^\perp = Nul\ A \text{ and } (Col\ A)^\perp = Nul\ A^T.$$