Math 300

Section 5.4 Eigenvectors and Linear Transformations

Let V be an n-dimensional vector space, let W be an m-dimensional vector space, and let T be any linear transformation from V to W. To associate a matrix with T, choose (ordered) bases $\mathcal{B}$ and $\mathcal{C}$ for V and W, respectively.

Let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be the basis $\mathcal{B}$ for V. If $\mathbf{x} = r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n$, then

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

and

$$T(\mathbf{x}) = T(r_1\mathbf{b}_1 + \cdots + r_n\mathbf{b}_n) = r_1T(\mathbf{b}_1) + \cdots + r_nT(\mathbf{b}_n).$$

This gives

$$[T(\mathbf{x})]_{\mathcal{C}} = r_1[T(\mathbf{b}_1)]_{\mathcal{C}+\cdots+r_n[T(\mathbf{b}_n)]_{\mathcal{C}}}$$

which implies

$$[T(\mathbf{x})]_{\mathcal{C}} = M[\mathbf{x}]_{\mathcal{B}}$$

where

$$M = [[T(\mathbf{b}_1)]_{\mathcal{C}} \ \cdots \ [T(\mathbf{b}_n)]_{\mathcal{C}}].$$

The matrix M is a matrix representation of T, called the matirx for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Theorem Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$.