**Math 300**

**Section 3.1** Introduction to Determinants

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$, denoted $detA$, is the sum of $n$ terms of the form $\pm a_{1j} detA_{1j}$, with plus and minus signs alternating, where $A_{ij}$ denotes the submatrix formed by deleting the $i$th row and $j$th column of $A$. In symbols,

$$\begin{aligned} detA &= a_{11} detA_{11} - a_{12} detA_{12} + \cdots + (-1)^{n+1} a_{1n} detA_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} detA_{1j} \end{aligned}$$

Given $A = [a_{ij}]$, the $(i, j)$-cofactor of $A$ is the number $C_{ij}$ given by

$$C_{ij} = (-1)^{i+j} detA_{ij}.$$

**Theorem** The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column. The expansion across the $i$th row using the cofactors is

$$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

The cofactor expansion down the $j$th column is

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

The plus or minus sign in the $(i, j)$-cofactor follows the checkerboard pattern:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \ddots & & \end{bmatrix}$$

**Theorem** If $A$ is a triangular matrix, then $detA$ is the product of the entries on the main diagonal of $A$.