Math 300

Section 2.9 Dimension and Rank

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \cdots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$) or the $\mathcal{B}$-coordinate vector of $\mathbf{x}$.

The transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate mapping. The coordinate mapping is a one-to-one onto linear transformation from the subspace H to $\mathbb{R}^m$, where m is the number of elements in the basis $\mathcal{B}$. A one-to-one onto linear transformation is called an isomorphism.

The dimension of a nonzero subspace H, denoted by $dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

The rank of a matrix A, denoted by $rank A$, is the dimension of the column space of A.

The Rank Theorem If a matrix A has n columns, then $rank A + dim Nul A = n$.

The Basis Theorem Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Invertible Matrix Theorem (Revamped)

Let A be an $n \times n$ matrix. Then the following are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

(m) The columns of A form a basis of $\mathbb{R}^n$.

(n) $ColA = \mathbb{R}^n$.

(o) $dimColA = n$.

(p) $rankA = n$.

(q) $NulA = \{\mathbf{0}\}$.

(r) $dimNulA = 0$.