Math 300

Section 2.3 Characterizations of Invertible Matrices

The Invertible Matrix Theorem (IMT)

Let A be an $n \times n$ matrix. Then the following are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

(g) The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(j) There is an $n \times n$ matrix C such that $CA = I_n$.

(k) There is an $n \times n$ matrix D such that $AD = I_n$.

(l) A^T is an invertible matrix.

Invertible Linear Transformations

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. In this case, we say $S = T^{-1}$.

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function for which $S(T(\mathbf{x})) = \mathbf{x}$ and $T(S(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.