Math 300

Section 1.9 The Matrix of a Linear Transformation

Theorem Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \ \ \forall \mathbf{x} \in \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix

$$A = [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)].$$

A is called the standard matrix for the linear transformation T.

Existence and Uniqueness Questions

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at least one $\mathbf{x}$ in $\mathbb{R}^n$.

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if each $\mathbf{b}$ in the range of T is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$.

Summarizing Chapter 1

Existence Questions

Let A be an $m \times n$ matrix. Then the following are equivalent:

1. The equation $A\mathbf{x} = \mathbf{b}$ is consistent for all vectors $\mathbf{b}$ in $\mathbb{R}^m$.
2. The columns of A span $\mathbb{R}^m$.
3. The matrix A has a pivot in each row.
4. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is onto.

Uniqueness Questions

Let A be an $m \times n$ matrix. Then the following are equivalent:

1. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
2. The system of equations with matrix equation $A\mathbf{x} = \mathbf{0}$ has no free variables.
3. The columns of A are linearly independent.
4. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.