Math 300

Section 1.7 Linear Independence

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly independent if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = 0$$

has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$. The set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly dependent if there exist $c_1, c_2, \cdots, c_p$ not all zero with

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = 0.$$

Notes

1. The columns of a matrix A are linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
2. The set $\{\mathbf{v}_1\}$ is linearly independent if and only if $\mathbf{v}_1 \neq \mathbf{0}$.
3. If the zero vector $\mathbf{0}$ is an element in a set of vectors, then that set must be linearly dependent.
4. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if one of the vectors is a scalar multiple of the other.
5. Suppose that $\mathbf{v}_1 \neq \mathbf{0}$ and that $p \geq 2$. Then the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ is linearly dependent if and only if some vector $\mathbf{v}_j$ with $j > 1$ is a linear combination of vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{j-1}\}$; that is, if and only if $\mathbf{v}_j$ with $j > 1$ is a linear combination of vectors preceding it in the set.
6. If a set of vectors contains more vectors that there are entries in each vector, then the set must be linearly dependent. That is, the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p\}$ of vectors in $\mathbb{R}^n$ is linearly dependent if $p > n$.