Math 300

Section 1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

If A is an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ and $\mathbf{x}$ is a vector in $\mathbb{R}^n$, then the product $A\mathbf{x}$ is a vector in $\mathbb{R}^m$ that is the linear combination of the columns of A using the corresponding entries as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots \mathbf{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n$$

Theorem Given an $m \times n$ matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ and given a vector $\mathbf{b}$ in $\mathbb{R}^m$, then the matrix equation $A\mathbf{x} = \mathbf{b}$ has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots x_n\mathbf{a}_n = \mathbf{b}$$

which has the same solution set as the system of linear equations with augmented matrix

$$[\mathbf{a}_1\ \mathbf{a}_2\ \cdots \mathbf{a}_n\ \mathbf{b}] = [A\ \mathbf{b}]$$

Theorem Let A be an $m \times n$ matrix. The following are equivalent:

1. For all vectors $\mathbf{b}$ in $\mathbb{R}^m$, the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution.
2. The columns of A span $\mathbb{R}^m$.
3. The matrix A has a pivot position in each row.

Properties of the Matrix-Vector Product

If A is an $m \times n$ matrix, the vectors $\mathbf{u}, \mathbf{v}$ are in $\mathbb{R}^n$ and c is a scalar, then

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
2. $A(c\mathbf{u}) = c(A\mathbf{u})$