Math 300

Section 1.2 Row Reduction and Echelon Forms

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry (ie, the leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

Theorem Any nonzero matrix may be row reduced (ie, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form of any matrix is unique.

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

The Row Reduction Algorithm

Step 1 Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

Step 2 Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Step 3 Use row replacement operations to create zeros in all positions below the pivot.

Step 4 Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply Steps 1-3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

Step 5 Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

A basic variable is a variable in a linear system that corresponds to a pivot column in the coefficient matrix.

A free variable is any variable in a linear system that is not a basic variable.

Existence and Uniqueness Theorem A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column - that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \cdots 0 \ b] \text{ with b nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.