The Monty Hall Problem

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Contents

Chapter 1

Ancestral Monty

1.1 A Mathematician's Life

Like all professional mathematicians, I take it for granted that most people will be bored and intimidated by what I do for a living. Math, after all, is the sole academic subject about which people brag of their ineptitude. "Oh," says the typical well-meaning fellow making idle chit-chat at some social gathering, "I was never any good at math." Then he smiles sheepishly, secure in the knowledge that his innumeracy in some way reflects well on him. I have my world-weary stock answers to such statements. I resist the temptation to say something snide ("How were you at reading?"), or downright nasty ("Perhaps you're just dim,") and instead say, "Well, maybe you just never had the right teacher." That typically defuses the situation nicely.

It is the rare person who fails to see humor in assigning to me the task of dividing up a check at a restaurant. You know, because I'm a mathematician. Like the elementary arithmetic used in check division is some sort of novelty act they train you for in graduate school. I used to reply with "Dividing up a check is applied math. I'm a pure mathematician," but this elicits puzzled looks from those who thought mathematics was divided primarily into the courses they were forced to take in order to graduate versus the ones they could mercifully ignore. I find, "Better have someone else do it. I'm not good with numbers," works pretty well.

I no longer grow vexed by those who ask, with perfect sincerity, how folks continue to do mathematical research when surely everything has been figured out by now. My patience is boundless for those who assure me that their grade-school nephew is quite the little math prodigy. When a student, after absorbing a scintillating presentation of, say, the mean-value theorem, asks me with disgust what it is good for, I am no longer even tempted to give him the smack in the face he so richly deserves. Instead I pretend not to realize he is merely expressing contempt for any subject that calls for both hard work and abstract thought, and launch into a discourse about all of the practical benefits that accrue from an understanding of calculus. ("You know how when you flip a switch the lights come on? Ever wonder why that is? It's because some really smart scientists like James Clerk Maxwell knew lots of calculus and figured out how to apply it to the problem of taming electricity. Kind of puts your whining into perspective, wouldn't you say?) And upon learning that a mainstream movie has a mathematician character, I feel cheated if that character and his profession are presented with any element of realism.

(Speaking of which, you remember that 1966 Alfred Hitchcock movie *Torn Curtain*, the one where physicist Paul Newman goes to Leipzig in an attempt to elicit certain German military secrets? Remember the scene where Newman starts writing equations on a chalkboard, only to have an impatient East German scientist, disgusted by the primitive state of American physics, cut him off and finish the equations for him? Well, we don't do that. We don't finish each other's equations. And that scene in *Good Will Hunting* where emotionally troubled math genius Matt Damon and Fields Medalist Stellan Skarsgaard high-five each other after successfully performing some feat of elementary algebra? We don't do that either. And don't even get me started on Jeff Goldblum in *Jurassic Park* or Russell Crowe in *A Beautiful Mind*...)

I tolerate these things because for all the petty annoyances resulting from society's impatience with math and science, being a mathematician has some considerable compensating advantages. My professional life is roughly equal parts doing mathematics and telling occasionally interested undergraduates about mathematics, which if you like math (and I *really* like math) is a fine professional life indeed. There is the pleasure of seeing the raised eyebrows on people's faces when it dawns on them that since I am a mathematician I must have a PhD in the subject, which in turn means that I am very, very smart. And then there is the deference I am given when the conversation turns to topics of math and science (which it often does, when I am in the room). That's rather pleasant. Social conventions being what they are, it is quite rare that my opinion on number-related questions is challenged.

Unless, that is, we are discussing the Monty Hall problem.

In this little teaser we are asked to play the role of a game show contestant confronted with three identical doors. Behind one is a car, behind the other two are goats. The host of the show, referred to as Monty Hall, asks us to pick one of the doors. We choose a door, but do not open it. Monty now opens a door different from our initial choice, careful always to open a door he knows to conceal a goat. We stipulate that if Monty has a choice of doors to open, then he chooses randomly from among his options. Monty now gives us the options of either sticking with our original choice, or switching to the one other unopened door. After making our decision, we win whatever is behind our door. Assuming that our goal is to maximize our chances of winning the car, what decision should we make?

So simple a scenario! And apparently with a correspondingly simple resolution. After Monty eliminates one of the doors, you see, we are left with a mere two options. As far as we are concerned, these two options are equally likely to conceal the car. It follows that there is no advantage to be gained one way or the other from sticking or switching, and consequently it makes no difference what decision we make. How sneaky to throw in that irrelevant nonsense about Monty choosing randomly when he has a choice! Surely you did not expect your little mathematical mind games to work on one so perspicacious as myself!

Or so it usually goes.

The minutiae of working out precisely why that intuitive and plausible argument is nonetheless incorrect will occupy us in the next chapter. For now I will note simply that it takes a person of rare sangfroid to respond with patience and humility to being told that the correct answer is to switch doors. You can share with a college class the glories of the human intellect, the most beautiful theorems and sublime constructs ever to spring forth from three pounds of matter in a human skull, and they will dutifully jot it all down in their notes without a trace of passion. But tell them that you double your chances of winning by switching doors, and suddenly the swords are drawn and the temperature drops ten degrees.

That PhD with which they were formerly so impressed? Forgotten! The possibility that they have overlooked some subtle point in their knee-jerk reaction to the problem? Never crosses their mind! They will explain with as much patience as they can muster, as though they were now the teacher and I the student, that *it makes no difference* what door you chose originally, or how Monty chose his door to open. It matters only that *just two doors remain* after Monty does his thing. Those doors, and this is the really important part, ***have an equal probability of being correct!***. And when I stubbornly refuse to accept their cogent logic, when I try to explain instead that there is, indeed, relevance to the fact that Monty follows a particular procedure in choosing his door, the chief emotion quickly shifts from anger to pity.

My remarks thus far may have given the impression that I find this reaction annoying. Quite the contrary, I assure you. My true emotion in these situations is delighted surprise. I have presented the problem to numerous college classes and in countless other social gatherings. No matter how many times I do so, I remain amazed by the ability of a mere math problem to awaken such passion and interest. The reason, I believe, is that the Monty Hall problem does not *look* like a math problem, at least not to people who think tedious symbol manipulation is what mathematics is really all about. The problem features no mathematical symbols, no excessively abstract terminology or ideas. Indeed, the problem can be explained to a middle-school student. The scenario it describes is one in which we can all imagine ourselves. And in such a situation, why should the egghead have any advantage over the normal folks?

1.2 Probability is Hard

The Monty Hall problem is a fine illustration of the difficulties most people encounter in trying to reason about uncertainty. Probabilistic reasoning is just not something that comes naturally. For myself, I remember the precise moment I came to realize that probability is hard. I was in high school, and my father proposed to me the following brainteaser (as we shall see throughout this book, my father often gave me puzzling problems to think about during my formative years): Imagine two ordinary, well-shuffled decks of cards on the table in front of you. Turn over the top card on each deck. What is the probability that at least one of those cards is the ace of spades?

Surely, I thought, we should reason that since a deck has fifty-two cards only one of which is the ace of spades, the probability of getting the ace on the first deck is $\frac{1}{52}$. The probability of getting the ace of spades on the second deck is likewise $\frac{1}{52}$. Since the two decks are independent of one another, the probability that at least one of those two cards was the ace of spades should be obtained by adding the two fractions together, thereby obtaining $\frac{1}{52} + \frac{1}{52} = \frac{1}{26}$. I gave my answer.

And that is when my father gave me the bad news. My answer was incorrect since I had not adequately considered the possibility of getting the ace of spades on both decks simultaneously. Why, though, should that be relevant? If we get the two aces simultaneously that is like a super victory! Getting them both may be overkill since only one ace was required, but it is not at all clear why that ought to alter my estimation of the probability.

Look a bit more closely, however, and you see that something is amiss. Imagine that instead of using fifty-two card decks, you instead remove the ace of spades and a joker from each deck. Now we have two small decks of two cards each. Let us repeat the experiment. There is now a probability of $\frac{1}{2}$ that the top card on the first deck is the ace of spades (and a corresponding probability of $\frac{1}{2}$ that the top card is a joker.) There is likewise a probability of $\frac{1}{2}$ that the top card of the second deck is the ace. Following our previous argument, we would now claim that the probability of obtaining the ace of spades on at least one of the two decks is given by $\frac{1}{2} + \frac{1}{2} = 1$, which would imply that we are certain to get an ace on at least one of the decks. This is plainly false, since there is a possibility of flipping up two jokers.

Turn now to the opposite extreme. Imagine that we have million card decks, with only one ace of spades in each deck. We repeatedly flip up the top card on each deck, careful to shuffle thoroughly between trials. Obtaining the ace of spades on either deck is an event of enormous rarity. When it happens, we expect to have to wait through another million trials before once again flipping up the ace on that deck. There is some small solace here. At least we expect to wait something less than that before flipping up the ace of spades on the other deck. Now imagine that the ace of spades come up simultaneously on both decks. In this event one of our precious ace flips has been wasted, no to happen again, on average, for another million trials. Flipping up the ace happens so infrequently that we can not afford such waste. The possibility of flipping up both aces simultaneously shows that we do not win as often as we think.

Let us try a more rigorous argument. Notice that there are $52 \times 52 = 2704$ different pairs of cards that can be formed by taking one card out of the first deck and one card out of the second deck (note that we are thinking of ordered pairs here, so that removing the two of spades from the first deck and the three of hearts from the second deck should be regarded as different from choosing the three of hearts from the first deck and the two of spades from the second). Now, there are 52 pairs where the ace of spades is the card chosen from the first deck (that is, the ace of spades from the first deck can be paired with any of the 52 cards from the second deck). There are likewise 52 pairs in which the card drawn from the second deck is the ace of spades. This makes a total of 104 out of 2704 pairs in which the ace of spades appears. Sadly, there is one pair that has been counted twice. Specifically, the pair in which both cards are the ace of spades has been double counted. Consequently, there are only 103 pairs of cards in which at least one of the cards is the ace of spades. This gives us a probability of $\frac{103}{2704}$, and that is our final answer. I would note that this fraction is equal to $\frac{1}{52} + \frac{1}{52} - \frac{1}{2704}$, which is the sum of the probabilities of getting the ace of spades on each deck individually, minus the probability of getting the ace on both decks. The justification for this formula will be presented at the appropriate time in this narrative.

This was a humbling experience for me. My father's scenario was superficially very simple. Just two decks of cards and a straightforward procedure. Yet a full understanding of what was going on required some careful analysis. Even after seeing the cold equations, the counterintuitive nature of the solution remains. That made quite an impression.

But for all of that, the Monty Hall problem looks at the two deck scenario and just laughs its head off. If the two deck problem struck you as frustratingly subtle, then there is a real danger that the Monty Hall problem will drive you insane. As much as I want people to read my book, I must advise you to consider turning back now.

1.3 The Perils of Intuition

It is customary for books about probability to try to persuade otherwise intelligent people that they are lousy when it comes to reasoning about uncertainty. There are numerous well-known examples to choose from in that regard. Since I see no reason to break from so fine a custom, I will present a few of my favorites below.

In presenting the Monty Hall problem to students I have found the common reactions to follow the well-known five stages of grief. There is Denial

(There is no advantage to switching.), Anger (How dare you suggest there is an advantage to switching!), Bargaining (It's really all a matter of perspective, so maybe we're both right.), Depression (Whatever. I'm probably wrong.) and Acceptance (Is this going to be on the test?). I figure I can speed that process along by showing you at the outset that your intuitions about probability are sometimes mistaken. Below are three of my favorites. (I realize, of course, that the first two examples in particular are so famous that it is possible you have heard them before. If that is the case then I apologize. But there are reasons they are classics!)

1.3.1 The Birthday Problem

Let us begin with an old chestnut known as the birthday problem. How many people do you need to assemble before the probability is greater than one half that some two of them have the same birthday? While you are pondering that, let me mention that I am not asking trick questions. You can safely assume there is no pair of identical twins among the people under discussion, you do not have to worry about leap years, every day is as likely as any other to be someone's birthday, and our birthdays consist of a month and a day with no year attached.

In the interest of putting a bit more space between our statement of the problem and its eventual solution, let me mention that the assumption that every day is as likely as any other to be someone's birthday is known to be unrealistic. For example, large numbers of children are conceived in the period between Christmas and New Year's, which leads to an unusually large number of children being born in August and September. Furthermore, many children are birthed via Caesarian section or induced labor, and these procedures are not generally scheduled for the weekends. This leads to an unusually large number of children being born on Mondays or Tuesdays (which is highly relevant for the teacher presenting this problem to a roomful of school children, since in that case most of the people in the room share a birth year.)

Back to the problem. A common answer is 183, since this number is the next integer larger than 365 divided by two. It is a reasonable answer, since with 183 people we can expect that more than half of the days in a normal year are represented. And if the question had been, "How many people do you need before obtaining a probability greater than one half that someone has the same birthday as you?" then 183 would be the correct answer.

But there's a big difference between someone in the crowd matching your birthday specifically, and some two people having the same birthday more generally. To see the distinction, imagine there are four people in the room. We shall call them Alice, Benjamin, Carol and Dennis. Then there are six ways of choosing two people from this group: (Alice, Benjamin), (Alice, Carol), (Alice, Dennis), (Benjamin, Carol), (Benjamin, Dennis) and (Carol, Dennis). Since only three of these pairs involve Alice, there are just three opportunities for someone to match Alice's birthday. This is in contrast to the six chances we have to get two arbitrary people matching a birthday.

From this we conclude that significantly less than 183 people are required. To determine the precise number, let us assume there are only two people, Alice and Benjamin. Then the probability is $\frac{364}{365}$ that they will have different birthdays. There are 364 days in the year that are not Alice's birthday, and each is as likely as any other to be Benjamin's birthday. If we add a third person, Carol, to the mix, there is a probability of $\frac{363}{365}$ that she will have a birthday different from Alice and Benjamin (because there are 363 days in the year that are the birthday of neither Alice nor Benjamin.) And if a fourth person now enters the room, the probability that his birthday is different from everyone else in the room will be $\frac{362}{365}$. The probability P that in a roomful of n people no two of them will have the same birthday is obtained by multiplying these numbers together, we obtain the formula

$$P = \left(\frac{364}{365}\right)\left(\frac{363}{365}\right)\left(\frac{362}{365}\right)\cdots\left(\frac{365-n+1}{365}\right).$$

This, recall, is the probability that we do not have two people with the same birthday. To answer the original question we must find the smallest value of

n for which $1 - P \geq \frac{1}{2}$. It turns out that $n = 23$ does the trick.

So just 23 people are needed to have a probability greater than $\frac{1}{2}$ of having two with the same birthday. With 23 people there are 253 pairs, which means 253 chances of getting a match. Remarkable. If you are curious, it turns out that with 88 people, the probability is greater than one half of having three people with the same birthday, while 187 people gives a probability greater than one half of four people having the same birthday. These facts are rather difficult to prove, however, and I will refer you to [57] for the full details.

1.3.2 False Positives in Medical Testing

Imagine a disease which afflicts roughly one out of every one thousand members of the population. There is a test for the disease, and this test is 95% accurate. It never gives a false negative; if it says you do not have the disease, then you do not have it. But 5% of the people who test positive for the disease are in reality disease-free. Let us suppose you have tested positive. What is the probability that you actually have the disease?

If you are inclined to answer that the probability is very high then your reasoning was probably that 95% is pretty close to certain and that is the end of the story. Overlooked in this argument, however, is the significance of the disease being very rare in the population. Only five percent of the positives returned by the test are false, but the disease only afflicts one tenth of one percent of the population. Your chance of having the disease is so small to begin with, that it should take something truly extraordinary to send you into a panic.

To see this, suppose that $100,000$ people take the test. Of these, we expect that 100 have the disease and $99,900$ are healthy. In that population of healthy people, we expect that 5%, or $4,995$, will receive a false positive. Since we are stipulating that the test has no false negatives, the one hundred people with the disease will also receive a positive test result. That makes a total of 100 true positives out of $5,095$ positive test results. Dividing 100 by $5,095$ gives us something just under 2%. Testing positive barely rates an eyebrow raise, much less a fit of panic. Indeed, even if the test were 99.9% accurate, it would still be just fifty-fifty that you have the disease (for in that case you would have .1% of $99,900$ people, or 99.9, receiving false positives. That makes 200 positive tests, of which 100 are accurate and 100 are not.)

This result is so surprising that we are in danger of thinking that the positive test result is essentially worthless as evidence of having the disease. This, however, would be the wrong conclusion. The positive test result changed our assessment of the probability of having the disease from .1% to 2%, a twenty-fold increase. It is simply that the probability of having the disease was so small to begin with, that even this increase is insufficient to make it seem likely. The tendency of people to ignore such considerations is referred to by psychologists and cognitive scientists as the "base-rate fallacy," (though I should mention there is some controversy over the extent to which people fall prey to this fallacy).

The source of the confusion lies in misapprehending the reference class to which the 5% applies. The 95% accuracy rate means that the huge majority of people who take the test get an accurate result. Most people will test negative, and will, in fact, be negative. These, however, are not the folks of interest to you upon testing positive. Instead you ought to concern yourself solely with the people who tested positive, and most of *those* people received a false result. The disease is so rare that a positive test result is far more likely to indicate an error than it is to indicate actual sickness.

Whole books get written exposing these sorts of pitfalls, and I recommend [77] as an especially good representative of the genre.

1.3.3 Thirty Percent Chance of Rain?

Sometimes the difficulty lies in the ambiguity of a probabilistic statement, as opposed to some error in reasoning. Take, for example, the weatherman's assertion that there is a thirty percent chance of rain tomorrow. What, exactly, does this mean?

In [32], a group of researchers polled residents of five different cities on precisely that question. The cities were New York, Amsterdam, Berlin, Milan and Athens. New York, you see, had introduced probabilistic weather forecasts in 1965. Amsterdam and Berlin did so in 1975 and the late eighties, respectively. In Milan they are used only on the internet, and in Athens they are not used at all. This provided some diversity in the level of exposure of the people of those cities to these sorts of forecasts.

Respondents were asked to assess which of the following three choices was the most likely, and which the least likely, to be the correct interpretation of the forecast:

1. It will rain tomorrow in thirty percent of the region.

2. It will rain tomorrow for thirty percent of the time.

3. It will rain on thirty percent of the days like tomorrow.

The correct answer is number three, though, in fairness, the wording is not quite right. A thirty percent chance of rain means roughly that in thirty percent of the cases when the weather conditions are like today, there was some significant amount of rain the following day. It was only in New York that a majority of respondents answered correctly. Option one was selected as most likely in each of the European cities.

The article goes on to discuss various cultural reasons for their findings, as well as some suggestions for how weather reporting bureaus can help clear up the confusion. For us, however, the take away message is that a probabilistic statement is ambiguous unless a clearly defined reference class is stipulated. There is nothing inherently foolish in any of the three interpretations above. A failure to pay attention to details, however, is a major source of error in probabilistic reasoning.

Along the same lines, the article [32] also contains the following amusing anecdote:

> A psychiatrist who prescribed Prozac to depressed patients used to inform that they had a $30\% - 50\%$ chance of developing a sexual problem such as impotence or loss of sexual interest.

> On hearing this, many patients became concerned and anxious. Eventually, the psychiatrist changed his method of communicating risks, telling patients that out of every ten people to whom he prescribes Prozac, three to five experience sexual problems. This way of communicating the risk of side effects seemed to put patients more at ease, and it occurred to the psychiatrist that he had never checked how his patients understood what a "30% – 50% chance of developing a sexual problem" means. It turned out that many had thought that something would go awry in 30% – 50% of their sexual encounters. The psychiatrist's original approach to risk communication left the reference class unclear: Does the percentage refer to a class of people (patients who take Prozac), to a class of events (a given person's sexual encounters), or to some other class? Whereas the psychiatrist's reference class was the total number of his patients who take Prozac, his patient's reference class was their own sexual encounters. When risks are solely communicated in terms of single-event probabilities, people have little choice but to fill in a class spontaneously based on their own perspective on the situation. Thus, single event probability statements invite a type of misunderstanding that is likely to go unnoticed.

And if *that* does not impress upon you the importance of thinking clearly about probability then I do not know what will!

1.4 The Legacy of Pascal and Fermat

Who is responsible for foisting on decent folks all of this subtlety and counter-intuition? There is a story in that, a small part of which will now be related.

The branch of mathematics devoted to analyzing problems of chance is known as probability theory. If it strikes you as odd that mathematics, a tool devised for explicating the regularities of nature, has any light to shed on unpredictable events, then rest assured you are in good company. Probability is a relative latecomer on the mathematical scene, and as recently as 1866 the British mathematician and philosopher John Venn (of Venn diagram fame) could write [92], without fear of being gainsaid,

> To many persons the mention of Probability suggests little else than the notion of a set of rules, very ingenious and profound rules no doubt, with which mathematicians amuse themselves by setting and solving puzzles.

Bertrand Russell expressed the paradox at the heart of probability by asking rhetorically, "How dare we speak of the laws of chance? Is not chance the antithesis of all law?" A resolution to this paradox begins with the observation that while individual events are frequently unpredictable, long series of the same kind of event can be a different matter entirely. The result of a single coin toss can not be predicted. But we can say with confidence that in a million tosses of a fair coin, the ratio of heads to tails will be very close to one.

That long-run frequencies can be stable where individual occurrences are not is a fact obvious to any gambler, and it is perhaps for this reason that probability emerged from a consideration of certain games of chance. Indeed, we now risk facing a different paradox. Since evidence of gambling goes back almost as far as human civilization itself, we might wonder why the mathematics of probability took so long to appear.

There were halting steps in that direction throughout the Middle Ages. This appears most notably in the work of Cardano and Galileo, both of whom noted that, in a variety of situations, there is insight to be gained from enumerating all of the possible outcomes from some particular chance scenario and assigning an equal probability to each. That said, it is fair to observe that probability in its modern form was born from a correspondence between the seventeenth century French mathematicians Blaise Pascal and Pierre de Fermat. The correspondence was the result of certain questions posed to Pascal by the Chevalier de Mere, a nobleman and gambler.

Especially significant was the “problem of points.” The general question is this: We have two players involved in a game of chance. The object of the game is to accumulate points. Each point is awarded in such a way that the players have equal chances of winning each point. The winner is the first player to reach a set number of points, and the prize is a pool of money to which both players have contributed equally. Suppose the game is interrupted prior to its completion. Given the score at the moment of the interruption, how ought the prize money be apportioned?

With the appearance of the word “ought” in the statement of the problem, we realize that this is not a question solely of mathematics. Some notion of fair play must be introduced to justify any proposed division. We will not delve into this aspect of things, preferring instead to use the principle that we know a fair division when we see one.

As a simple example, imagine the players are Alistair and Bernard. Points are awarded by the toss of a coin, with heads going to Alistair and tails going to Bernard. Let us say the winner is the first one to ten points, and the score is currently 8 for Alistair and 7 for Bernard. This is roughly the scenario pondered by Pascal and Fermat.

Fermat got the ball rolling by noting that the game will surely end after no more than four further tosses of the coin. This corresponds to sixteen scenarios, all of them equally likely. Since it is readily seen that eleven of these scenarios lead to victories for Alistair while a mere five lead to victories for Bernard, the prize money should be divided in the ratio of $11 : 5$, with the larger share going to Alistair.

Not much to gainsay there, but Pascal one-upped him by noting that enumerating all of the possibilities becomes tedious in a hurry, and can become effectively impossible for large numbers. What is needed is a general method for counting the number of scenarios in which each of the players win. He pictured a situation in which Alistair needed n points to win, while Bernard needed m points. In that case the game would end in no more than $n+m-1$ plays, for a total of 2^{n+m-1} possible scenarios. You can imagine listing all of these possible scenarios, recording A for an Alistair point and B for a point to Bernard. The result will be 2^{n+m-1} strings of A's and B's. Any string in which $n, n+1, n+2, \cdots, n+m-1$ plays come up in favor of Alistair will correspond to an Alistair victory. In modern notation, we would say that we seek the sum of the following binomial coefficients:

$$\binom{n+m-1}{n}, \binom{n+m-1}{n+1}, \binom{n+m-1}{n+2}, \cdots, \binom{n+m-1}{n+m-1}.$$

Pascal's name was already associated with such objects, as he had previously written extensively about them. In particular, he devoted considerable attention to what is now known as Pascal's triangle (though the existence of the triangle was known well before Pascal arrived on the scene.) The values of the binomial coefficients can be arranged in a triangle, the first five rows of which are as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Armed with this triangle, we find row $n+m$, and sum up the first m numbers we find there. This will give us the number of scenarios in which Alistair wins.

For example, in our original game we saw that Alistair needed two more points to win while Bernard needed three. Consequently, we have $n = 2$ and $m = 3$. It follows that the game will be over in no more than $2 + 3 - 1 = 4$ subsequent plays. To determine the number of scenarios in which Alistair wins, we go to row $n+m = 5$ of the triangle. We find the numbers $1, 4, 6, 4, 1$. The sum of the first m of these entries is $1 + 4 + 6 = 11$, which is precisely what Fermat found with his less elegant approach.

Nowadays these are the sorts of considerations that appear very early in any standard undergraduate course in probability or statistics. At the time, however, this was impressive stuff. It represents one of the first serious attempts to develop a calculus for probabilities, and the level of algebraic sophistication achieved by Pascal and Fermat went far beyond anything that had been seen previously. These basic principles prove to be adequate for solving the classical Monty Hall problem.

Were this intended to be a history of probability generally, I would here take note of the many people either contemporary with or prior to Pascal who contributed to the then nascent theory of probability. If you are interested in such a history, I recommend either Ian Hacking's *The Emergence of Probability* [40], or F. N. David's *Games, Gods, and Gambling* [17]. (David's book is especially interesting for her opinion, in defiance of the consensus view, that Pascal's contributions have been overrated, and that Fermat deserves more credit than he gets). But since my actual intention is to find someone to blame for the endless stream of counterintuitive probabilistic brainteasers with which generations of undergraduate math majors have been tormented, I will follow convention and blame the correspondence between Pascal and Fermat.

1.5 What Bayes' Wrought

Thus far we have been concerned primarily with the problem of inferring the effects of known causes. For example, given what we know about coins, what are we likely to observe if one is tossed multiple times? Likewise for the problem of points. We are given much regarding the structure of the game, and seek some reasonable conclusion as to how things are likely to proceed.

This sort of thinking, however, can be turned around. Sometimes we have known effects and wish to work backward to what caused them. Typically there are many plausible causes for given mysterious effects. We seek some statement about which of them is most likely to be correct. This is referred to as the problem of inverse probability, and it was pioneered by an eighteenth century British mathematician and Presbyterian minister named Thomas Bayes. His discussion of the problem appears in an essay published posthumously in 1764 entitled "An Essay Towards Solving a Problem in the Doctrine of Chances."

Upon resolving to include in this book some material on the history of probability, I thought it might be fun to read Bayes' essay. I was mistaken. Even with several modern commentaries to guide me, I found it largely impenetrable. Bayes' writing is frequently muddled and confusing, and you will search his essay in vain for anything that looks like what we now call Bayes theorem. (A special case of the theorem appears in Bayes' essay. The modern form of Bayes' theorem received its first careful formulation in the work of Laplace.) Add to this the inevitable difficulties that arise in trying to read technical papers written long ago, at a time when much modern terminology and notation had yet to arrive on the scene, and I would say you are better off with a modern textbook.

Bayes occupies a curious position in the history of mathematics. His name is today attached to a major school of philosophical thought on the nature of probability (more on that in Chapter Three). "Bayesianism" also refers to an influential view of proper statistical reasoning. Nevertheless, histories of probability can not seem to dismiss Bayes quickly enough. David's history of the early days of probability [17] contains not a single reference to him. Hacking mentions him only briefly in [40]. Isaac Todhunter's magisterial and still authoritative 1865 book *History of the Theory of Probability From the Time of Pascal to That of Laplace* [**?**] devotes a chapter to Bayes, but at six pages long it is the shortest in the book. It also quite critical of Bayes' work.

The mathematical details of Bayes' theorem will occupy us in Chapter Three. For now let us consider the more general question of how to update a prior probability assessment in the face of new evidence. In the Monty Hall problem, for example, let us assume that we initially choose door number one. Since the doors are assumed to be identical, we assign a probability of $\frac{1}{3}$ to this door. We now see Monty open one of the other two doors to reveal a goat. The question is whether our $\frac{1}{3}$ probability assignment ought to change in the light of this new information.

It will be useful, in pondering such situations, to change our perspective regarding the nature of probability. To this point we have behaved as though the point of probability was to discover the properties of certain real world objects. Assigning a probability of $\frac{1}{2}$ to the result of a coin toss was viewed as a statement about coins, for example. More specifically, it was a description of something coins tend to do when they are flipped a large number of times. This, however, is not the only way of viewing things. We might also think of probability assignments as representing our degree of belief in a given proposition. In this view, the assignment of $\frac{1}{2}$ to each possible result of a coin toss means that we have no basis for believing that one outcome is more likely than another. It is a statement about our beliefs, as opposed to a statement about coins.

We now ask for the variables affecting how we update our degree of belief in a proposition in the face of new evidence. One consideration, I suggest, is obvious. Our updated assignment will depend in part on our prior assignment. Scientists have a saying that extraordinary claims require extraordinary evidence. This captures the insight that if we initially view a proposition as exceedingly unlikely, it will take impressive evidence indeed to make it suddenly seem likely.

The next consideration is less obvious. If A is the proposition whose probability we are trying to assess, and B is the new evidence, we want to know how tightly correlated B occurrences are with A occurrences. That is, we need to know how likely it is that B will occur given that A is true. If we assess this probability as very high, then the occurrence of B will increase our confidence in the truth of A. Perhaps we are on a jury in a criminal trial. We learn that a hair found at the scene of the crime provides a DNA match with the defendant. Since the probability that such evidence will be found given that the suspect is guilty is quite high, the DNA match would tend to increase our confidence in the guilt of the defendant. But now suppose we learn that the suspect has an unbreakable alibi for the time of the crime. Since the probability that the defendant would have such an alibi given that he is guilty is quite low, this revelation would decrease our confidence in the guilt of the defendant.

This, however, can not be the end of the story. It is possible that B is the sort of thing that happens frequently regardless of whether or not A is true. In such a situation we would assess that the probability of B given that A is true is high not because of any particular connection between A and B, but simply because B is something that is very likely to happen regardless. The finding that B is very likely to happen diminishes its relevance as evidence for A. What matters, then, is not just how likely it is that B will occur given that A is true. Rather, we seek the ratio of this probability to the probability that B will occur barring any assumption about A.

In other words, B should be viewed as strong evidence in support of A if B is something that is likely to occur if A is true, but unlikely to occur if A is false. Let us suppose our defendant has no plausible reason for being present at the scene of the crime. Then we might say that finding his DNA at the scene is likely to happen only if he is guilty, and the DNA match is strong evidence for the prosecution. But if it turns out that the crime was committed in a place the defendant often frequents for entirely innocent reasons, then the DNA match is likely to occur independent of any assumption about his guilt. In this case, the DNA match constitutes weak evidence indeed.

Bayes' theorem takes these vague intuitions and turns them into a precise formula for updating prior probability assessments. It will be our constant companion through most of this book.

1.6 The Bertrand Box Paradox

The Monty Hall problem in its modern form goes back to 1975, and I assure you we will arrive at *that* little matter soon enough. For sixteen years prior to that it was traveling incognito as the Three Prisoners Problem. That will be the subject of the next section.

Annoying brainteasers in conditional probability, however, have a far longer history, and there is one little bagatelle of sufficient importance to rate a mention in this chapter. It is nowadays referred to as the Bertrand Box Paradox, in honor of French mathematician Joseph Bertrand. It appeared in his 1889 book *Calcul des Probabilites* (Calculus of Probabilities) as follows (see [67] for a useful discussion of Bertrand's thinking):

> Three boxes are identical in external appearance. The first box contains two gold coins, the second contains two silver coins, and the third contains a coin of each kind, one gold and one silver. A box is chosen at random. What is the probability that it contains the unlike coins?

If your first instinct was that the problem is trivial, but then worried that if it were *really* trivial I would not have included it, then rest assured that this is one of the few places in the book where you may trust your first instinct. It really is trivial. Let us denote the box containing the two gold coins by B_{gg}, the box containing the two silver coins by B_{ss}, and the box containing one of each by B_{gs}. Since the boxes are identical they have equal probabilities of being chosen. And since there is only one box in which the coins are different, we find the probability of having chosen B_{gs} to be $\frac{1}{3}$. I should mention that this question appeared on page two of the book, and was intended merely to illustrate the idea of enumerating a set of equally likely possibilities.

Bertrand, however, did not leave things here. He went on to wonder how one ought to react to the following argument: Let us suppose we choose one drawer at random and remove one of the coins without looking at it. Regardless of the coin we choose, there are only two possibilities: the remaining coin in the box is either gold or it is silver. It is, therefore, either like or unlike the unexamined coin we have just removed. That makes two possibilities, each equally likely, and in only one of them are the coins different. It would seem that the removal of the coin caused our probability to jump from $\frac{1}{3}$ to $\frac{1}{2}$.

This argument is plainly fallacious, since the mere removal of one unidentified coin in no way increases our knowledge of the coin in the other drawer. Bertrand reasoned that the fallacy lay in assuming that the two possibilities (the coin being either like or unlike the coin we removed) were equally likely. In fact, since there are two chests in which the coins are the same and only one in which they are different, it is self-evident that like coins are more probable than unlike coins. If we find, for example, that the coin we removed was gold, then the other coin is more likely to be gold than silver.

To see this, note that since the chosen coin is gold, we are removing from consideration the possibility that we chose B_{ss}. If we reached into B_{gg}, then the probability of removing a gold coin is equal to 1. But if we reached into B_{gs}, there is a probability of just $\frac{1}{2}$ of removing the gold. It follows that we are twice as likely to remove a gold coin having chosen B_{gg} than we would have been had we chosen B_{gs}. And since these probabilities must sum to 1, we find that the other coin will be silver with probability $\frac{1}{3}$ (and will be gold with probability $\frac{2}{3}$, just as we found previously.

Bertrand intended this as a cautionary tale of what happens when you are too cavalier in assigning equal probabilities to events. Lest you find this point too trivial to bother with, I assure you that some very competent mathematicians throughout history have managed to bungle it. In a famous example, the French mathematician Jean le Rond d'Alembert once argued that the probability of tossing at least one head in two tosses of a coin is $\frac{2}{3}$. He argued there were only three possibilities: we could get a head on the first toss or, barring that, get a tail on the first toss and then a head on the second toss. We can represent these as H, TH, TT. He then treated these possibilities as equiprobable, from which his answer follows. Of course, a proper analysis would note that the coin comes up heads on the first toss half the time, while the scenario TH happens one fourth of the time. Summing these possibilities leads to the correct probability of $\frac{3}{4}$ for getting heads at least once in two tosses of a coin.

A more modern form of Bertrand's problem begins with the same set-up, and asks for the probability that the second coin in the box is gold given that one coin was removed at random and seen to be gold. Stated thusly we have a standard problem in inverse probability. We initially assign an equal probability to each box. The new information is that our chosen box contains a gold coin. How ought we to revise our probability assessments? Bertrand's argument shows that updating the probability of an event given new information requires considering the probability of obtaining the information given the event. For example, the probability of having chosen B_{gs} given that we removed a gold coin depends in part on the probability of removing a gold coin having chosen B_{gs}. As we have mentioned, this insight lies at the heart of Bayes' Theorem.

The similarity between this scenario and the Monty Hall problem is clear. In both scenarios an initial selection of three equiprobable options is narrowed to two in the light of new information. There is a tendency, in both cases, to assign an equal probability to the two remaining scenarios, but this must be resisted. Understanding Bertrand's problem is a useful first step towards resolving the Monty Hall scenario.

1.7 The Three Prisoners

In a 1959 column for *Scientific American*, Martin Gardner wrote

> Charles Sanders Pierce once observed that in no other branch of mathematics is it so easy for experts to blunder as in probability theory. History bears this out. Leibniz thought it just as easy to throw 12 with a pair of dice as to throw 11. Jean le Rond d'Alembert, the great 18-th century French mathematician, could not see that the results of tossing a coin three times are the same as tossing three coins at once, and he believed (as many amateur gamblers persist in believing) that after a long run of heads, a tail is more likely.

In light of the explosion over the Monty Hall problem that would occur just over three decades later, these words seem downright prophetic. That notwithstanding, our interest in this section resides in a particular brain-teaser, presented by Gardner as follows:

> A wonderfully confusing little problem involving three prisoners and a warden, even more difficult to state unambiguously, is now making the rounds. Three men - A, B and C - were in separate cells under sentence of death when the governor decided to pardon one of them. He wrote their names on three slips of paper, shook the slips in a hat, drew out one of them and telephoned the warden, requesting that the name of the lucky man be kept secret for several days. Rumor of this reached prisoner A. When the warden made his morning rounds, A tried to persuade the warden to tell him who had been pardoned. The warden refused.
>
> "Then tell me," said A, "the name of one of the others who will be executed. If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."
>
> "But if you see me flip the coin." replied the wary warden, "you'll know that you're the one pardoned. And if you see that I don't flip a coin, you'll know it's either you or the person I don't name."
>
> "Then don't tell me now," said A. "Tell me tomorrow morning."
>
> The warden, who knew nothing about probability theory, thought it over that night and decided that if he followed the procedure suggested by A, it would give A no help whatever in estimating his survival chances. So next morning he told A that B was going to be executed.
>
> After the warden left, A smiled to himself at the warden's stupidity. There were now only two equally probable elements

> in what mathematicians like to call the "sample space" of the problem. Either C would be pardoned or himself, so by all the laws of conditional probability, his chances of survival had gone up from $\frac{1}{3}$ to $\frac{1}{2}$.
>
> The warden did not know that A could communicate with C, in an adjacent cell, by tapping in code on a water pipe. This A proceeded to do, explaining to C exactly what he had said to the warden and what the warden had said to him. C was equally overjoyed with the news because he figured, by the same reasoning used by A, that his own survival chances had also risen to $\frac{1}{2}$.
>
> Did the two men reason correctly? If not, how should each calculate his chances of being pardoned?

This, surely, is the Monty Hall problem in all but name. Simply replace the three prisoners with three doors, the pardon with the car, the prisoners to be executed with the doors concealing the goats, and the warden with Monty Hall. Now, I realize that it is a perilous thing for an historical researcher to declare that X is the first instance of Y. If it subsequently turns out that Y was lurking in some obscure corner of the academic literature, you can be certain that some overeducated braggart will delight in pointing out the fact. That risk notwithstanding, I would mention that I have dozens of professional references discussing this problem, and not one of them cites anything earlier than Gardner's column as its source. In personal correspondence Gardner was gracious enough to tell me he did not find the problem in any older published source, but rather that he heard the problem from various acquaintances. My own considerable researches have likewise failed to turn up any older reference. So, I am calling it. Gardner's 1959 column is the first published instance of the Monty Hall problem, or at least of something formally equivalent to it.

Gardner presented the correct solution, that A will be pardoned with probability $\frac{1}{3}$ while C's chances have improved to $\frac{2}{3}$, in the following issue of the magazine [30]. He offered two arguments: first, by enumerating the sample space, and alternatively by making an analogy to a situation with vastly more prisoners. Since the next chapter is devoted entirely to a consideration of such arguments, we will not discuss them here. We should also note the care with which Gardner stated the problem. In particular, he was explicit that the warden chooses randomly when given a choice of prisoners to name. This detail is essential to a proper solution of the problem, but it is often omitted in casual statements of it.

That said, in the spirit of showing just how difficult it can be to provide a truly pristine analysis of the problem, we can point to one unfortunate bit of phrasing in Gardner's presentation of the problem's solution. He writes, "Regardless of who is pardoned, the warden can give A the name of a man, other than A, who will die. The warden's statement therefore had no influence on A's survival chances; they continue to be $\frac{1}{3}$." Writing in [22], psychologist Ruma Falk points out the difficulty with this sentence, "Both parts of that sentence are correct, just the adverb "therefore", used here with conjunctive force, is inapt." The conclusion that A's probability does not change does not follow merely from the fact that the warden can always reveal the name of one of A's fellow prisoners. Rather, it is a consequence of the precise method used by the warden in deciding which name to reveal. We will revisit this point in the next chapter.

After Gardner's column, the three prisoner's problem accumulated quite a literature, much of it providing an eerie parallel to the Monty Hall fracas that would erupt in the early nineties. Statistician Fred Mosteller included it as problem thirteen in his *Fifty Challenging Problems in Probability with Solutions* [62] in 1965. In presenting the solution, he remarked that this problem attracts far more mail from readers than any other. Biologist John Maynard Smith, after presenting the problem in his 1968 book *Mathematical Ideas in Biology* [84], remarked, "This should be called the Serbelloni problem since it nearly wrecked a conference on theoretical biology in the summer of 1966; it yields at once to common sense or to Bayes' theorem." I certainly accept the latter part of that disjunction, but accumulated painful experience has left me dubious regarding the former.

As a case in point I would mention a statement made by Nicholas Falletta in his otherwise excellent 1983 book *The Paradoxicon* [23]. After presenting the three prisoners problem, Falletta writes, "Prisoner A reasoned that since he was now certain that B would die then his chances for survival had improved from $\frac{1}{3}$ to $\frac{1}{2}$ and, indeed, they had!" Alas, Falletta did not explain how he came to this conclusion. He also neglected, in his statement of the problem, to tell his readers how the warden went about choosing which name to reveal. If he was envisioning the usual assumption, that the warden chooses randomly when given a choice, then we must regard Falletta's statement as simply incorrect. It is possible to imagine procedures the warden could follow that would justify Falletta's statement, but then these details needed to be spelled out. We will have more to say about this in the chapters ahead.

For now, let us note simply that it is a rare problem indeed that has been immortalized in verse. The prisoner's problem can claim that distinction, courtesy of mathematician Richard Bedient in [8].

The Prisoner's Paradox Revisited

Awaiting the dawn sat three prisoners wary
A trio of brigands named Tom, Dick and Mary
Sunrise would signal the death knoll of two
Just one would survive, the question was who

Young Mary sat thinking and finally spoke
To the jailer she said, "You may think this a joke.
But it seems that my odds of surviving 'til tea,
Are clearly enough just one out of three.

But one of my cohorts must certainly go,
Without question, that's something I already know.

Telling the name of one who is lost,
Can't possibly help me. What could it cost?"

That shriveled old jailer himself was no dummy,
He thought, "But why not?" and pointed to Tommy.
"Now it's just Dick and I," Mary chortled with glee.
"One in two are my chances, and not one in three!"

Imagine the jailer's chagrin, that old elf.
She'd tricked him, or had she? Decide for yourself.

1.8 Let's Make a Deal

In 1963 the television game show *Let's Make A Deal* premiered on American television. In its initial run it lasted until 1977. In each episode the host, Monty Hall, engaged in various games with members of his audience. These games had a number of formats, but the general principle was typically the same. Players had to decide between definitely winning a small prize, or gamble on some probability of winning a greater prize.

In one game, which often served as the show's climax, contestants were shown three identical doors and were told that behind one of them was a car, while the other two concealed goats. Recall that in the abstract version of the problem considered here and in the next chapter, the game unfolds as follows: The contestant chooses but does not open a door. Monty now opens a door he knows to conceal a goat, choosing randomly when he has a choice. He then gives the player the options of sticking with his original choice or switching to the other unopened door. The contestant makes his choice and wins whatever is behind his door.

This is not, however, how things unfolded on the show. Typically, if the contestant initially chose a goat the door was opened immediately and the game ended on the spot. But if the contestant chose the car, Monty opened one of the remaining doors and gave the contestant the option of switching.

This option was sometimes accompanied by an offer of cash money from Monty not to make the switch (if this offer was accepted, the player took the cash and went home, not opening any of the doors). If the player insisted on switching nonetheless, Monty would sometimes offer still more money, at times reaching as high as a few thousand dollars. This was a highly effective psychological ploy to make it seem that the car was behind the other remaining door. Of course, a devoted watcher of the show might have picked up on Monty's skullduggery, but that does not seem to have happened too often in practice.

Things do not get mathematically interesting until we stipulate that Monty always opens a goat-concealing door and always gives the option of switching. This might explain why it would be another dozen years until the term "Monty Hall problem," would enter the mathematical literature.

1.9 The Birth of the Monty Hall Problem

In February 1975, the academic journal *The American Statistician* published a letter to the editor from Steve Selvin, then a mathematician at the University of California at Berkeley, proposing the following exercise in probability [78]. Given its considerable historical significance, we reproduce it in full:

> It is "Let's Make a Deal" - a famous TV show starring Monte Hall.
>
> MONTY HALL: One of the three boxes labeled A, B and C contains the keys to that new 1975 Lincoln Continental. The other two are empty. If you choose the box containing the keys, you win the car.
>
> CONTESTANT: Gasp!
>
> MONTY HALL: Select one of these boxes.
>
> CONTESTANT: I'll take box B.
>
> MONTY HALL: Now box A and box C are on the table and here is box B (contestant grips box B tightly). It is possible the

car keys are in that box! I'll give you $ for the box.

CONTESTANT: No, thank you.

MONTY HALL: How about $200?

CONTESTANT: No!

AUDIENCE: No!!

MONTY HALL: Remember that the probability of your box containing the keys to the car is $\frac{1}{3}$ and the probability of your box being empty is $\frac{2}{3}$. I'll give you $500.

AUDIENCE: No!!

CONTESTANT: No, I think I'll keep this box.

MONTY HALL: I'll do you a favor and open one of the remaining boxes on the table (he opens box A). It's empty! (Audience: applause). Now either box C or your box B contains the car keys. Since there are two boxes left, the probability of your box containing the keys is now $\frac{1}{2}$. I'll give you $1000 cash for your box.

WAIT!!!!

Is Monty right? The contestant knows that at least one of the boxes on the table is empty. He now knows that it was box A. Does this knowledge change his probability of having the box containing the keys from $\frac{1}{3}$ to $\frac{1}{2}$? One of the boxes on the table has to be empty. Has Monty done the contestant a favor by showing him which of the two boxes was empty? Is the probability of winning the car $\frac{1}{2}$ or $\frac{1}{3}$?

CONTESTANT: I'll trade you my box B for the box C on the table.

MONTY HALL: That's Weird!!

HINT: The contestant knows what he is doing.

The logic verifying the correctness of the contestant's strategy is then presented in the form of a table enumerating all the possibilities. It seemed straightforward enough, since a quick inspection revealed that in six out of nine possible scenarios, the contestant would win the car by switching. Considering the venue, a high level journal read primarily by professional statisticians, you would have expected a raised eyebrow or two and little more. But this is the Monty Hall problem we are discussing, and it has the power to make otherwise intelligent people take leave of their senses.

Selvin's letter was published in February. By August, Professor Selvin was back in the letters page with a follow-up [79]. His letter was entitled, "On the Monty Hall Problem," and we note with great fanfare this earliest known occurrence of that phrase in print. Selvin noted that he received a number of letters in response to his earlier essay taking issue with his proposed solution. He went on to present a second argument in defense of his conclusion, this time a more technical one involving certain formulas from conditional probability.

Especially noteworthy is the following statement from Selvin's follow-up: "The basis to my solution is that Monty Hall knows which box contains the keys and when he can open either of two boxes without exposing the keys, he chooses between them at random." In writing this he had successfully placed his finger on the two central points of the problem. Alter either of those assumptions and the analysis can become even more complex, as we shall see in the chapters to come.

In an amusing coda to this story, Selvin notes in his follow-up that he had received a letter from Monty Hall himself:

> Monty Hall wrote and expressed that he was not a "student of statistics problems" but "the big hole in your argument is that once the first box is seen to be empty, the contestant cannot exchange his box." He continues to say, "Oh, and incidentally, after one [box] is seen to be empty, his chances are no longer 50/50 but remain what they were in the first place, one out of three. It just seems to the contestant that one box having been eliminated, he stands a better chance. Not so."

It would seem that Monty Hall was on top of the mathematical issues raised by his show. It is a pity that more mathematicians were not aware of Selvin's lucid analysis. They might thereby have spared themselves considerable public embarrassment.

Another decade and a half would go by before the Monty Hall problem really left its mark on the mathematical community. While the three prisoners problem continued to feature prominently in professional articles from various disciplines, the Monty Hall problem was mostly dormant throughout the eighties. There were a few exceptions. Statisticians Persi Diaconis and Sandy Zabell used it in a 1986 paper [**?**] to discuss various approaches to the problem of inverse probability. It also appeared in 1987, when Barry Nalebuff [64] presented it in the inaugural problem section of the academic journal *Economic Perspectives*. For the most part, however, the problem was still flying decidedly below the radar during this time.

1.10 L'Affaire *Parade*

It is September 9, 1990. President Samuel Doe of the small African nation of Liberia is assassinated by rebel forces as part of one of the bloodiest Civil Wars that continent would ever see. Untied States President George Bush and Russian President Mikhail Gorbachev present a joint statement protesting the illegal occupation of Kuwait by Iraqi military forces. Tennis star Pete Sampras won the first of his record-setting fourteen Grand Slam tennis championships by defeating fellow American Andre Agassi in the finals of the U.S. Open. The uncut version of Stephen King's horror masterpiece *The Stand* rests at number five on the *New York Times* bestseller list.

And Marilyn vos Savant, a Q and A columnist for *Parade* magazine, responds to the following question from reader Craig Whitaker of Columbia, Maryland [94]:

> Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car, behind the others, goats.

> You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to pick door number 2?" Is it to your advantage to switch your choice of doors?

Ms. vos Savant replied as follows:

> Yes, you should switch. The first door has a $\frac{1}{3}$ chance of winning, but the second door has a $\frac{2}{3}$ chance. Here's a good way to visualize what happened. Suppose there are a *million* doors, and you pick door number 1. Then the host, who knows what's behind the doors and will always avoid the one with the prize, opens them all except door number $777,777$. You'd switch to that door pretty fast, wouldn't you?

With this exchange we open one of the strangest chapters in the history of mathematics.

The Oxford University biologist Richard Dawkins once responded to an extremely hostile, and badly misinformed, review of one of his books [18] by writing, "Some colleagues have advised me that such transparent spite is best ignored, but others warn that the venomous tone of her article may conceal the errors in its content. Indeed, we are in danger of assuming that nobody would dare to be so rude without taking the elementary precaution of being right in what she said." He might as well have been discussing the response to vos Savant's proposed solution.

Mind you, I can understand why someone, even a professional mathematician, would be caught out by the Monty Hall problem. It is genuinely counter-intuitive, even for people with serious training in probability and statistics. As we shall see in the next chapter, no less a personage than Paul Erdos, one of the most famous mathematicians of the twentieth century, not only got the problem wrong, but stubbornly refused to accept the correct answer for quite some time. The prominent Stanford University mathematician Persi Diaconis once said of the Monty Hall problem [88], "I can't remember what my first reaction to it was because I've known about it for so many years. I'm one of many people who have written papers about it. But I do know that my first reaction has been wrong time after time on similar problems. Our brains are just not wired to do probability problems very well, so I'm not surprised there were mistakes."

But if getting it wrong is understandable, being snotty and condescending about it is not. In a follow-up column on December 2, vos Savant shared some of the choicer items from her mailbox. I have not troubled here to reproduce the names of the correspondents, since it is not my intention to embarrass anyone. The morbidly curious can check out vos Savant's book [94]. Suffice it to say that the correspondents below were all mathematicians:

> Since you seem to enjoy coming straight to the point, I'll do the same. In the following question and answer, you blew it! Let me explain. If one door is shown to be a loser, that information changes the probability of either remaining choice, *neither of which has any reason to be more likely,* to $\frac{1}{2}$. As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and in the future being more careful.

And:

> You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain. After the host reveals a goat, you now have a one-in-two chance of being correct. Whether you change your selection or not, the chances are the same. There is enough mathematical illiteracy in this country, and we don't need the world's highest I.Q. propagating more. Shame!

And:

> Your answer to the question is in error. But if it is any consolation, many of my academic colleagues have also been stumped by this problem.

In replying, vos Savant rightly stuck to her original answer. This time she opted for the approach of enumerating the sample space, which really ought to have ended the discussion. It did not.

On February 17, 1991, vos Savant revisited the problem yet again. Once more she shared the musings of some of her more obnoxious correspondents:

> May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?

And:

> I have been a faithful reader of your column, and I have not, until now, had any reason to doubt you. However, in this matter (for which I do have expertise), your answer is clearly at odds with the truth.

And:

> You are utterly incorrect about the game-show question, and I hope this controversy will call some public national attention to the serious national crisis in mathematical education. If you can admit your error, you will have contributed constructively towards the solution of a deplorable situation. How many irate mathematicians are needed to get you to change your mind?

And:

> You made a mistake, but look at the positive side. If all those PhD's were wrong, the country would be in some very serious trouble.

Looks like the country is in serious trouble.

Marilyn vos Savant's answer this time introduced two important nuances into the discussion. First, she stated more explicitly than previously that it is crucial to assume that Monty always opens a losing door. Relaxing that assumption in any way changes the problem completely.

Second, she proposed that mathematics classes across the country put her proposed solution to the test. The Monty Hall scenario is easily simulated, making it possible to run through a large number of trials in a relatively short period of time. By having one segment of the class follow an "always switch" strategy while having the other electing to "always stick," it becomes a straightforward matter to see who wins more frequently. In mathspeak we would say that vos Savant was proposing the question be resolved via a Monte Carlo simulation.

Many folks took her up on this suggestion, leading to a fourth column on the subject. The letters inspiring this one were of a considerably different tone:

> In a recent column, you called on math classes around the country to perform an experiment that would confirm your response to a game show problem. My eighth-grade classes tried it, and I don't really understand how to set up an equation for your theory, but it definitely does work. You'll have to help rewrite the chapters on probability.

And:

> Our class, with unbridled enthusiasm, is proud to announce that our data support your position. Thank you so much for your faith in America's educators to solve this.

And:

> I must admit I doubted you until my fifth-grade math class proved you right. All I can say is WOW!

And:

> After considerable discussion and vacillation here at the Los Alamos National Laboratory, two of my colleagues independently programmed the problem, and in 1 million trials, switching paid off 66.7 percent of the time. The total running time on the computer was less than one second.

Most amusing of all, one letter writer had the audacity to write: "Now 'fess up. Did you really figure all this out, or did you get help from a mathematician?"

So vos Savant was vindicated. Nothing succeeds like success, and to anyone actually playing the game multiple times it quickly becomes clear that switching is the way to go.

This story has a curious footnote. Shortly after vos Savant wrote her last column on the Monty Hall problem, she received the following letter:

> A shopkeeper says she has two new baby beagles to show you, but she doesn't know whether they're male, female, or a pair. You tell her that you want only a male, and she telephones the fellow who's giving them a bath. "Is at least one a male?" she asks him. "Yes!" she informs you with a smile. What is the probability that the *other* one is a male?

She replied with the correct answer that the probability is one out of three. There are three ways to have a pair of puppies in which one is male, you see, and these scenarios are equally likely. Listing the puppies in the order of their birth, they could be male/female, female/male or male/male. Since it is only the last of these scenarios in which "the other puppy" is male, we arrive at our answer. This problem, in various forms, is itself a classic problem in probability, and we shall have more to say about it in a later chapter.

By now you should not be surprised to learn that a storm of angry correspondence ensued, most of them lecturing vos Savant about how the sex of each puppy is entirely independent of the sex of the other one. A family could have five sons in a row, but the chances are still fifty-fifty that the next child will be a daughter.

This is not in dispute. It also is not what was asked. Had the problem said, "The older puppy is a male. What is the probability that the younger puppy is a male also?" then the answer would surely be one half. As it is, however, there is no first puppy or second puppy.

1.11 The *Am. Stat.* Exchange

The humiliation dealt to the mathematical community in the wake of *L'Affaire Parade* could not be ignored. Countless simulations made clear the fact that Marilyn vos Savant had answered the question correctly, suggesting that her rather intemperate correspondents had a lot of crow to eat. That notwithstanding, it was possible that some portion of the blame could still be laid at her feet. While her answer was surely correct, perhaps her reasoning left something to be desired. This tactic took its most pointed form in an exchange of letters in the academic journal *The American Statistician* [58], [59].

I have already reproduced vos Savant's first solution to the problem. Let us now consider her subsequent attempts. Her second gambit was the following:

> My original answer is correct. But first, let me explain why your answer is wrong. The winning chances of 1/3 on the first choice can't go up to 1/2 just because the host opens a losing door. To illustrate this, let's say we play a shell game. You look away, and I put a pea under one of three shells. Then I ask you to put your finger on a shell. The chances that your choice contains a pea are 1/3, agreed? Then I simply lift up an empty shell from

> the remaining two. As I can (and will) do this regardless of what you've chosen, we've learned nothing to allow us to revise the chances on the shell under your finger.
>
> The benefits of switching are readily proven by playing through the six games that exhaust all the possibilities. For the first three games, you chose number 1 and "switch" each time, for the second three games, you choose number 1 and "stay" each time, and the host always opens a loser. Here are the results:

She now produced a table listing the various scenarios. In the interests of conserving space, I will note that she described three scenarios which can be described as AGG, GAG and GGA depending on the location of the car. These scenarios are equally likely. If we now assume that you always choose door one and that Monty only opens goat-concealing doors, we see than in two of the three scenarios you win by switching. She then concluded:

> When you switch, you win 2/3 of the time and lose 1/3, but when you don't switch, you only win 1/3 of the time and lose 2/3. You can try it yourself and see.
>
> Alternatively, you can actually play the game with another person acting as the host with three playing cards – two jokers for the goats and an ace for the prize. However, doing this a few hundred times to get statistically valid results can get a little tedious, so perhaps you can assign it as extra credit – or for punishment! (*That'll* get their goats!)

vos Savant gave a still more detailed treatment in her third column. I will beg your indulgence as I present a lengthy excerpt. It is necessary for fully understanding what happened next.

> So let's look at it again, remembering that the original answer defines certain conditions, the most significant of which is that *the host always opens a losing door on purpose.* (There's no way

he can always open a losing door by chance!) Anything else is a different question.

The original answer is still correct, and the key to it lies in the question *"Should you switch?"* Suppose we pause at this point, and a UFO settles down onto the stage. A little green woman emerges, and the host asks her to point to one of the two unopened doors. The chances that *she'll* randomly choose the one with the prize are 1/2, all right. But that's because she lacks an advantage the *original* contestant had – the help of the host. (Try to forget any particular television show.)

When you first choose door number 1 from three, there's a 1/3 chance that the prize is behind that one and a 2/3 chance that it's behind one of the others. *But then the host steps in and gives you a clue.* If the prize is behind number 2, the host shows you number 3, and if the prize is behind number 3, the host shows you number 2. So when you switch, you win if the prize is behind number 2 *or* number 3. *You win either way!* But if you *don't* switch, you win only if the prize is behind door number 1.

And as this problem is of such intense interest, I'll put my thinking to the test with a nationwide experiment. This is a call to math classes all across the country. Set up a probability trial exactly as outlined below and send me a chart of all the games along with a cover letter repeating just how you did it, so we can make sure the methods are consistent.

One student plays the contestant, and another, the host. Label three paper cups number 1, number 2, and number 3. While the contestant looks away, the host randomly hides a penny under a cup by throwing a die until a one, two, or three comes up. Next, the contestant randomly points to a cup by throwing a die the same way. Then the host purposely lifts up a losing cup from the two unchosen. Lastly, the contestant "stays" and lifts up his

> original cup to see if it covers the penny. Play "not switching" two hundred times and keep track of how often the contestant wins.
>
> Then test the other strategy. Play the game the same way until the last instruction, at which point the contestant instead "switches" and lifts up the cup *not* chosen by anyone to see if it covers the penny. Play "switching" two hundred times, also.

Certainly vos Savant's arguments are not mathematically rigorous, and we can surely point to places where her phrasing might have been somewhat more precise. Her initial argument based on the million door case is pedagogically effective, but mathematically incomplete (as we shall see). And there was a subtle shift from the correspondent's initial question, in which the host always opens door three, to the listing of the scenarios given by vos Savant, in which it was assumed only that the host always opens a goat-concealing door.

But for all of that it seems clear that vos Savant successfully apprehended all of the major points of the problem, and explained them rather well considering the forum in which she was writing. Her intent was not to provide an argument of the sort a mathematician would regard as definitive, but rather to illuminate the main points at issue with arguments that would be persuasive and comprehensible. In this she was successful.

Four people who were less impressed were mathematicians J. P. Morgan, N. R. Chaganty, R. C. Dahiya and M. J. Doviak (MCDD). Writing in *The American Statistician*, they presumed to lay down the law regarding vos Savant's treatment of the problem. After quoting the original question as posed by vos Savant's correspondent, they write:

> Marilyn vos Savant, the column author and reportedly holder of the world's highest I.Q., replied in the September article, "Yes, you should switch. The first door has a 1/3 chance of winning, but the second door has a 2/3 chance." She then went on to

> give a dubious analogy to explain the choice. In the December article letters from three PhD's appeared saying that vos Savant's answer was wrong, two of the letters claiming that the correct probability of winning with either remaining door is 1/2. Ms. vos Savant went on to defend her original claim with a false proof and also suggested a false simulation as a method of empirical verification. By the February article a full scale furor had erupted; vos Savant reported, "I'm receiving thousands of letters nearly all insisting I'm wrong.... Of the letters from the general public, 92% are against my answer; and of letters from universities, 65% are against my answer." Nevertheless, vos Savant does not back down, and for good reason, as, given a certain assumption, her answer is correct. Her methods of proof, however, are not.

Rather strongly worded, wouldn't you say? And largely unfair, for reasons I have already discussed. Indeed, continuing with their lengthy essay makes clear that their primary issue with vos Savant is her shift from what they call the "conditional problem," as posed by her correspondent (in which it is stipulated that the contestant always chooses door one and the host always opens door three) to the "unconditional problem," in which we stipulate only that after the contestant chooses a door, the host opens one of the goat-concealing doors. She did, indeed, make this shift, but this was hardly the point at issue between vos Savant and her angry letter-writers.

The authors go on to provide an interesting and useful probabilistic analysis of the problem, but that is not our interest here. At present we are interested in the human drama that surrounded the appearance of the problem in vos Savant's column. With that in mind, let us ponder vos Savant's response to MCDD. After quoting the original question and her original answer, vos Savant writes,

> It should be understood by an academic audience that this problem, written for a popular audience, was not intended to be

> subject to strong attempts at misinterpretation. If it had been, it would have been a page long. While it may be instructionally constructive to purposely focus on semantic issues here, it is surely intellectually destructive to imply that it reflects negatively on the perspicacity of the writer involved.

And later,

> Nearly all of my critics understood the intended scenario, and few raised questions of ambiguity. I personally read nearly three thousand letters (out of many more thousands that ultimately arrived) and found virtually every reader, from university lectern to kitchen table, insisting simply that because two options remained, the chances were even.
>
> I would continue, but I find my initial annoyance flagging and will instead devote my energy to confounding the editorial staff at *Parade* once again, especially as it has occurred to me that the authors have clearly at least found a way to provoke me to sit down and write a response when other readers have failed to do so. Frankly, after seeing this problem analyzed on the front page of the *New York Times* and now creating a similar stir in England, I have given up on getting the facts across properly and have decided simply to sit back and amuse myself with the reading of it all.

Zing!

MCDD responded to this. They had the audacity to begin with, “We are surprised at the tone of vos Savant’s reply.” It is unclear what tone they were expecting in response to their bellicose and condescending essay. They then repeated the main points from their earlier essay, emphasizing that the problem vos Savant discussed was not precisely the problem laid out in the initial question. This point is not at issue, but what vos Savant discussed was surely what was intended. Even if it was not, vos Savant made it quite clear what problem she *was* discussing. Seen in that light, MCDD ought not to have said that her arguments were wrong and contained technical errors when they meant simply that she had altered the problem slightly from what was originally stated. In fairness, MCDD do moderate their tone later on, writing, "None of this diminishes the fact that vos Savant has shown excellent probabilistic judgment in arriving at the answer 2/3, where, to judge from the letters in her column, even member of our own profession failed."

I have belabored this incident for two reasons. The first is to capture for you the heat and emotion that has characterized so many discussions of this issue, even in otherwise staid, professional outlets. I would hardly be doing my job as a chronicler of all things related to the Monty Hall problem if I did otherwise.

The other is to illustrate what I perceive as an occupational hazard among mathematicians. Specifically, the desire always to be the smartest person in the room. In my experience, this sad tendency is especially prevalent when interacting with non-mathematicians. The relish with which MCDD declare vos Savant's arguments to be wrong is both palpable, and completely uncalled for.

They, at least, were mathematically correct in their substantive points. The motives of the letter writers whose hectoring and arrogant missives have deservedly earned them a place in the mathematical hall of shame are even more incomprehensible. What could possibly make people think it is acceptable to write with such a tone over a mere exercise in probability theory?

1.12 The Aftermath

Writing in the magazine *Bostonia* [71], cognitive scientist Massimo Piatelli-Palmarini aptly described the Monty Hall problem by writing, "...no other statistical puzzle comes so close to fooling all the people all the time...The phenomenon is particularly interesting precisely because of its specificity, its reproducibility, and its immunity to higher education."

In a front-page article for the Sunday *New York Times* on July 21, 1991, John Tierney summed things up as follows:

> Since she gave her answer, Ms. vos Savant estimates she has received 10,000 letters, the great majority disagreeing with her. The most vehement criticism has come from mathematicians and scientists, who have alternated between gloating at her, ("You are the goat!") and lamenting the nation's innumeracy.
>
> Her answer – that the contestant should switch doors – has been debated in the halls of the Central Intelligence Agency and the barracks of fighter pilots in the Persian Gulf. It has been analyzed by mathematicians at the Massachusetts Institute of Technology and computer programmers at Los Alamos National Laboratory in New Mexico. It has been tested in classes from second grade to graduate level at more than 1,000 schools across the country.

Since the initial fracas erupted, the Monty Hall problem has accumulated a formidable technical literature. It would seem that researchers from a wide variety of disciplines found something of interest within its simple scenario. Mathematicians and statisticians hashed out the probabilistic issues raised by the problem and its variants [3], [10], [14], [33], [53], [75], [77], [81]. Philosophers found connections between the Monty Hall problem and various longstanding problems in their own discipline [6], [7], [11], [15], [44], [52], [60]. Physicists devised quantum mechanical versions of the problem [16], [100]. Cognitive scientists and psychologists tried to determine why, exactly, people have so much trouble with this problem [1], [34], [36], [37], [38], [39], [45], [46], [51]. Economists pondered the relevance of the Monty Hall problem to the problems of human decision making [48], [70], [72], [83], [90]. These are just a few representative citations. There are many others.

We will have occasion to look at much of this research in the pages ahead, but this introduction has gone on long enough. It is time to do some math!

1.13 Appendix: Dignity in Problem Statements

Steve Selvin gave the first ever published presentation of the Monty Hall problem in the form of a play. In doing so he inaugurated a disturbing trend in published statements of the problem. Apparently believing the scenario is insufficiently confusing when presented flat-out as a teaser in probability, many authors feel the need to embed it within some larger bit of melodrama. Here is an example, taken from [56] (all emphases in original):

> *Announcer:* And now ... the game show that mathematicians argue about ... LET"S MAKE A DEAL. Here's your genial host, Monty Hall! [Applause]
>
> *Monty:* Hello, good evening, and welcome! Now let's bring up our first contestant. It's ... YOU! Come right up here. Now, you know our rules. Here are three doors, numbered 1, 2, and 3. Behind one of these doors is a beautiful new PONTIAC GRAN HORMONISMO!
>
> *Audience:* Oooh! Aahh!
>
> *Monty:* Behind the other two are WORTHLESS GOATS!
>
> *Audience:* [Laughter]
>
> *Goats:* Baah!
>
> *Monty:* Now, you're going to choose one of those doors. Then I'm going to open one of the other doors with a goat behind it, and show you the goat. Then I'll offer you this deal: if you stick with the door you've chosen, you can keep what's behind it, plus $100. If instead you chose the remaining unopened door, you can keep what's behind it. Now choose one door.
>
> *Audience:* Pick 3! No, 1! 2!
>
> *You:* Um, oh well, I guess I'll pick...3.
>
> *Monty:* Okay. Now our beautiful host Charleen will open door number 2. Inside that door, as you can see, is a WORTHLESS

> GOAT. You can keep what's behind your door 3 plus $100, or you can make a deal and switch for whatever's behind door 1. While we take our commercial break, you should decide: do you wanna MAKE A DEAL??

It's the line where the goats say "Baah!" that really makes you feel like you're there.

In the course of describing the relevance of the Monty Hall problem to bridge players [55] (details in Chapter Nine), Phil Martin serves up the following presentation:

> "Behind one of these three doors," shouts Monty Hall, "is the grand prize, worth one hundred thousand dollars. It's all yours – if you pick the right door."
>
> "I'll take door number one," you say.
>
> "Let's see what's behind door number – No! Wait a minute!" says Monty Hall. "Before we look, I'll offer you *twenty thousand dollars*, sight unseen, for whatever's behind door number one."
>
> "No! No!" shouts the audience.
>
> "Of course not, you say. "Even assuming the booby prizes are worth nothing, the expected value of my choice is thirty-three and a third thousand dollars. Why should I take twenty thousand?"
>
> "All right, says Monty Hall. "But before we see what you've won, let's take a look behind *door number two*!"
>
> Door number two opens to reveal one of the booby prizes: a date in the National Open Pairs with Phil Martin. You and the audience breathe a sigh of relief.
>
> "I'll give you one last chance," says Monty Hall. "You can have *forty* thousand dollars for what's behind door number one."
>
> "No, no!" shouts the audience.
>
> "Sure," you say.

I really must protest these cheap theatrics. The problem has all it can handle getting itself stated with sufficient clarity to be mathematically tractable. Embedding it in a skit only makes it harder to parse, and typically, as in the two examples above, leads to important assumptions not being spelled out. So knock it off!

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