Section 3.6 Existence and Nonexistence of Periodic Solutions

Definition The $\omega-$limit set for an orbit $\phi(t, x_0)$ with right maximal interval of existence $[0, \infty)$ is defined to be

$$\{z \in \mathbb{R}^n : \text{ there is a sequence } t_k \to \infty \text{ so that } \phi(t_k, x_0) \to z\}.$$

The $\alpha-$limit set for an orbit $\phi(t, x_0)$ with left maximal interval of existence $(-\infty, 0]$ is defined to be

$$\{z \in \mathbb{R}^n : \text{ there is a sequence } t_k \to -\infty \text{ so that } \phi(t_k, x_0) \to z\}.$$

Definition A cycle is a nonconstant periodic solution of $x' = f(x)$. If a cycle is the $\omega-$limit set or the $\alpha-$limit set of a distinct orbit, then the cycle is called a limit cycle. If a limit cycle is the $\omega-$limit set of every nearby orbit, then the limit cycle is said to be stable.

Poincare-Bendixson Theorem Consider

$$x' = f(x, y)$$
$$y' = g(x, y).$$

If $\phi(t, x)$ is a bounded orbit for $t \geq 0$ and W is its $\omega-$limit set, then either W is a cycle, or for each $y \in W$, the $\omega-$limit set of $\phi(t, y)$ is a set of one or more equilibrium points.

Theorem Inside any cycle there has to be an equilibrium point

$$x' = f(x, y)$$
$$y' = g(x, y).$$

Definition We define the divergence of the vector field

$$F(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

by

$$divF(x, y) = f_x(x, y) + g_y(x, y).$$

Definition A domain $D \subseteq \mathbb{R}^2$ is said to be a simply connected domain provided it is connected and for any simple closed curve C in D the interior of C is a subset of D.

Bendixson-Dulac Theroem Assume there is a continuously differentiable function $\alpha(\cdot, \cdot)$ on a simply connected domain $D \subseteq \mathbb{R}^2$ such that the

$$div[\alpha(x, y)F(x, y)]$$

is either always positive or always negative on D. Then the system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

does not have a cycle in D.