Section 3.4 Stability of Nonlinear Systems

Definition Let $x_0 \in \mathbb{R}^n$ and $r > 0$.

(i) The open ball centered at x_0 with radius r is the set

$$B(x_0, r) \equiv \{x \in \mathbb{R}^n : \|x - x_0\| < r\}.$$

(ii) Let x_0 be an equilibrium point for $x' = f(x)$. Then x_0 is stable if for each ball $B(x_0, r)$, there is a ball $B(x_0, s)$ (here $s \leq r$) so that if $x \in B(x_0, s)$, then $\phi(t, x)$ remains in $B(x_0, r)$ for $t \geq 0$.

(iii) If, in addition to the conditions in part (ii), there is a ball $B(x_0, p)$ so that for each $x \in B(x_0, p)$, $\phi(t, x) \to x_0$ as $t \to \infty$, then x_0 is asymptotically stable.

Definition If $V : \mathbb{R}^n \to \mathbb{R}$ has partial derivatives with respect to each component of x, then we define the gradient of V to be the $1 \times n$ matrix function

$$gradV(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1}(x) & \frac{\partial V}{\partial x_2}(x) & \cdots & \frac{\partial V}{\partial x_n}(x) \end{bmatrix}.$$

Definition Let x_0 be an equilibrium point for $x' = f(x)$. A continuously differentiable function V defined on an open set $U \subset \mathbb{R}^n$ with $x_0 \in U$ is called a Lyapunov function for $x' = f(x)$ on U provided $V(x_0) = 0$, $V(x) > 0$ for $x \neq x_0$, $x \in U$, and

$$gradV(x) \cdot f(x) \leq 0,$$

for $x \in U$. If $gradV(x) \cdot f(x) < 0$ for $x \in U$, $x \neq x_0$, then V is called a strict Lyapunov function for $x' = f(x)$ on U.

Lyapunov's Stability Theorem If V is a Lyapunov function for $x' = f(x)$ on an open set U containing an equilibrium point x_0, then x_0 is stable. If V is a strict Lyapunov function, then x_0 is asymptotically stable.

Definition A set S is said to be positively invariant for system $x' = f(x)$ if for each $x_0 \in S$, $\phi(t, x_0) \in S$, for $t \in [0, \omega)$.

Theorem If V is a Lyapunov function for $x' = f(x)$ on a bounded open set U, then for any constant $c > 0$ such that the set $\{x \in U : V(x) \leq c\}$ is closed in $\mathbb{R}^n$, this set is positively invariant.

LaSalle Invariance Theorem Let V be a Lyapunov function for $x' = f(x)$ on a bounded open set U containing an equilibrium point x_0. If $c > 0$ is a constant so that $S := \{x : V(x) \leq c\}$ is a closed set in $\mathbb{R}^n$, and if there is no $x \neq x_0$ in S for which $V(\phi(t, x))$ is constant for $t \geq 0$, then for all $x \in S$, $\phi(t, x) \to x_0$ as $t \to \infty$.