Section 3.1 Introduction to Autonomous Systems

Existence/Uniqueness Theorem Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. Then for each $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$, the initial value problem (IVP)

$$x' = f(x), \;\; x(t_0) = x_0,$$

has a solution x. Furthermore, x has a maximal interval of existence (α, ω), where $-\infty \leq \alpha < t_0 < \omega \leq \infty$. If $\alpha > -\infty$, then $\lim_{t \to \alpha^+} \|x(t)\| = \infty$, and if $\omega < \infty$, then $\lim_{t \to \omega^-} \|x(t)\| = \infty$. If, in addition, f has continuous first-order partial derivatives with respect to $x_1, x_2, \cdots, x_n$, on $\mathbb{R}^n$, then the above IVP has a unique solution.

Theorem For each t in its maximal interval of existence, $\phi(t, x)$ is continuous as a function of x.

Definition If x is a solution of

$$x' = f(x), \;\; x(t_0) = x_0$$

on its maximal interval of existence (α, ω), then $\{x(t) : \alpha < t < \omega\} \subseteq \mathbb{R}^n$ is called an orbit or trajectory of $x' = f(x), \;\; x(t_0) = x_0$.

Theorem The following hold:

(i) Assume $x(t)$ satisfies $x' = f(x), \;\; x(t_0) = x_0$ on an interval (c, d). Then for any constant h, $x(t - h)$ satisfies $x' = f(x), \;\; x(t_0) = x_0$ on the interval $(c + h, d + h)$.

(ii) If two orbits of $x' = f(x), \;\; x(t_0) = x_0$ have a common point, then the orbits are identical.

Definition We say that x_0 is an equilibrium point of $x' = f(x), \;\; x(t_0) = x_0$ provided

$$f(x_0) = 0.$$

An equilibrium point x_0 is a one-point orbit since $x(t) = x_0$ is a constant solution of $x' = f(x), \;\; x(t_0) = x_0$.