Section 2.5 Floquet Theory

Jordan Canonical Form If A is an $n \times n$ constant matrix, then there is a nonsingular $n \times n$ constant matrix P so that $A = PJP^{-1}$, where J is a block diagonal matrix of the form

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix},$$

where either J_i is the 1×1 matrix $J_i = [\lambda_i]$ or

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_i & 1 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix},$$

$1 \le i \le k$, and the λ_i's are the eigenvalues of A.

Log of a Matrix If C is an $n \times n$ nonsingular matrix, then there is a matrix B such that

$$e^B = C.$$

Floquet's Theorem If Φ is a fundamental matrix for the Floquet system $x' = A(t)x$, where the matrix function A is continuous on $\mathbb{R}$ and has minimum positive period ω, then the matrix function Ψ defined by $\Psi(t) := \Phi(t + \omega), t \in \mathbb{R}$, is also a fundamental matrix. Furthermore there is a nonsingular, continuously differentiable $n \times n$ constant matrix B (possibly complex) so that

$$\Phi(t) = P(t)e^{Bt},$$

for all $t \in \mathbb{R}$.

Definition Let Φ be a fundamental matrix for the Floquet system $x' = A(t)x$. Then the eigenvalues μ of

$$C := \Phi^{-1}(0)\Phi(\omega)$$

are called the Floquet multipliers of the Floquet system $x' = A(t)x$.

Theorem Let $\Phi(t) = P(t)e^{Bt}$ be as in Floquet's theorem. Then x is a solution of the Floquet system $x' = A(t)x$ iff the vector function y defined by $y(t) = P^{-1}(t)x(t), t \in \mathbb{R}$ is a solution of $y' = By$.

Theorem Let $\mu_1, \mu_2, \cdots, \mu_n$ be the Floquet multipliers of the Floquet system $x' = A(t)x$. Then the trivial solution is

(i) globally asymptotically stable on $[0, \infty)$ iff $|\mu_i| < 1, 1 \le i \le n$;

(ii) stable on $[0, \infty)$ provided $|\mu_i| \le 1, 1 \le i \le n$ and whenever $|\mu_i| = 1$, μ_i is a simple eigenvalue;

(iii) unstable on $[0, \infty)$ provided there is an $i_0, 1 \le i_0 \le n$, such that $|m_{i_0}| > 1$.

Theorem The number μ_0 is a Floquet multiplier of the Floquet system $x' = A(t)x$ iff there is a nontrivial solution x such that

$$x(t+\omega) = \mu_0 x(t),$$

for all $t \in \mathbb{R}$. Consequently, the Floquet system has a nontrivial periodic solution of period ω iff $\mu_0 = 1$ is a Floquet multiplier.

Theorem Assume $\mu_1, \mu_2, \cdots, \mu_n$ are the Floquet multipliers of the Floquet system $x' = A(t)x$. Then

$$\mu_1 \mu_2 \cdots \mu_n = e^{\int_0^\omega tr[A(t)]dt}.$$