**Section 2.3** The Matrix Exponential Function

**Definition** Let $A$ be an $n \times n$ constant matrix. Then we define the matrix exponential function $e^{At}$ as the solution of the IVP

$$X' = AX, \quad X(0) = I,$$

where $I$ is the $n \times n$ identity matrix.

**Cayley-Hamilton Theorem** Every $n \times n$ constant matrix satisfies its characteristic matrix.

**Putzer Algorithm for Finding $e^{At}$** Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the (not necessarily distinct) eigenvalues of the matrix $A$. Then

$$e^{At} = \sum_{k=0}^{n-1} p_{k+1}(t) M_k,$$

where $M_0 := I$,

$$M_k := \prod_{i=1}^{k} (A - \lambda_i I),$$

for $1 \le k \le n$ and the vector function $p$ defined by

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{bmatrix},$$

for $t \in \mathbb{R}$, is the solution of the IVP

$$p' = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} p, \quad p(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

**Theorem** Assume $A$ and $B$ are $n \times n$ constant matrices. Then

(i) $\dfrac{d}{dt} e^{At} = A e^{At}$, for $t \in \mathbb{R}$,

(ii) $\det[e^{At}] \neq 0$, for $t \in \mathbb{R}$ and $e^{At}$ is a fundamental matrix for $x' = Ax$,

(iii) $e^{At} e^{As} = e^{A(t+s)}$, for $t, s \in \mathbb{R}$,

(iv) $\{e^{At}\}^{-1} = e^{-At}$, for $t \in \mathbb{R}$ and, in particular,

$$\{e^{A}\}^{-1} = e^{-A},$$

(v) if $AB = BA$, then $e^{At} B = B e^{At}$, for $t \in \mathbb{R}$ and, in particular,

$$e^{A} B = B e^{A},$$

(vi) if $AB = BA$, then $e^{At} e^{Bt} = e^{(A+B)t}$, for $t \in \mathbb{R}$ and, in particular,

$$e^{A} e^{B} = e^{A+B},$$

(vii) $e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^k\frac{t^k}{k!} + \cdots$ , for $t \in \mathbb{R}$,

(viii) if $P$ is a nonsingular matrix, then $e^{PBP^{-1}} = Pe^B P^{-1}$.

**Variation of Constants** Assume that $A$ is an $n \times n$ continuous matrix function on an interval $I$, $b$ is a continuous $n \times 1$ vector function on $I$, and $\Phi$ is a fundamental matrix for $x' = A(t)x$. Then the solution of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I$ and $x_0 \in \mathbb{R}^n$, is given by

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)b(s)ds,$$

for $t \in I$.

**Corollary** Assume $A$ is an $n \times n$ constant matrix and $b$ is a continuous $n \times 1$ vector function on an interval $I$. Then the solution $x$ of the IVP

$$x' = Ax + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I, x_0 \in \mathbb{R}^n$ is given by

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}b(s)ds,$$

for $t \in I$.

**Stability Theorem** Assume $A$ is an $n \times n$ constant matrix.

(i) If $A$ has an eigenvalue with positive real part, then the trivial solution of $x' = Ax$ is unstable on $[0, \infty)$.

(ii) If all the eigenvalues of $A$ with zero real parts are simple (multiplicity one) and all other eigenvalues of $A$ have negative real parts, then the trivial solution is stable on $[0, \infty)$.

(iii) If all the eigenvalues of $A$ have negative real parts, then the trivial solution of $x' = Ax$ is globally asymptotically stable on $[0, \infty)$.