Section 2.2 The Vector Equation $x' = A(t)x$

Definition We say that the constant $n \times 1$ vectors $\psi_1, \psi_2, \cdots, \psi_k$ are linearly dependent provided there are constants $c_1, c_2, \cdots, c_k$, not all zero, such that

$$c_1\psi_1 + c_2\psi_2 + \cdots + c_k\psi_k = 0,$$

where 0 denotes the $n \times 1$ zero vector. Otherwise we say that these k constant vectors are linearly independent.

Theorem Assume we have exactly n constant $n \times 1$ vectors

$$\psi_1, \psi_2, \cdots, \psi_n$$

and C is the column matrix $C = [\psi_1\psi_2 \cdots \psi_n]$. Then $\psi_1, \psi_2, \cdots, \psi_n$ are linearly dependent iff $\det(C) = 0$.

Definition Assume the $n \times 1$ vector functions $\phi_1(t), \phi_2(t), \cdots, \phi_k(t)$ are defined on an interval I. We say that these k vector functions are linearly dependent on I provided there are constants $c_1, c_2, \cdots, c_k$, not all zero, such that

$$c_1\phi_1(t) + c_2\phi_2(t) + \cdots c_k\phi_k(t) = 0,$$

for all $t \in I$. Otherwise we say that these k vector functions are linearly independent on I.

Theorem The linear vector differential equation $x' = A(t)x$ has n linearly independent solutions on I, and if $\phi_1, \phi_2, \cdots, \phi_n$ are n linearly independent solutions on I, then

$$x = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n,$$

for $t \in I$, where $c_1, c_2, \cdots, c_n$ are constants, is a general solution of $x' = A(t)x$.

Definition Let A be a given $n \times n$ constant matrix and let x be a column unknown n-vector. For any number λ the vector equation

$$Ax = \lambda x$$

has the solution $x = 0$ called the trivial solution of the vector equation. If λ_0 is a number such that the vector equation $Ax = \lambda_0 x$ has a nontrivial solution x_0, then λ_0 is called an eigenvalue of A and x_0 is called a corresponding eigenvector. We say λ_0, x_0 is an eigenpair of A.

Theorem If λ_0, x_0 is an eigenpair for the constant $n \times n$ matrix A, then

$$x(t) = e^{\lambda_0 t}x_0, \quad t \in \mathbb{R},$$

defines a solution x of

$$x' = Ax$$

on $\mathbb{R}$.

Theorem If $x = u + iv$ is a complex vector-valued solution of $x' = A(t)x$, where u, v are real vector-valued functions, then u, v are real vector-valued solutions of $x' = A(t)x$.

Theorem Assume A is a continuous $n \times n$ matrix function on an interval I and assume that Φ defined by

$$\Phi(t) = [\phi_1(t), \phi_2(t), \cdots, \phi_n(t)], \quad t \in I,$$

is the $n \times n$ matrix function with columns $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$. Then Φ is a solution of the matrix differential equation $X' = A(t)X$ on I iff each column ϕ_i is a solution of the vector differential equation $x' = A(t)x$ on I for $1 \leq i \leq n$. Furthermore, if Φ is a solution of the matrix differential equation $X' = A(t)X$, then

$$x(t) = \Phi(t)c$$

is a solution of the vector differential equation $x' = A(t)x$ for any constant $n \times 1$ vector c.

Existence-Uniqueness Theorem Assume A is a continuous matrix function on an interval I. Then the IVP

$$X' = A(t)X, \quad X(t_0) = X_0,$$

where $t_0 \in I$ and X_0 is an $n \times n$ constant matrix, has a unique solution X that is a solution on the whole interval I.

Definition Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

Then we define the <u>trace</u> of $A(t)$ by

$$\text{tr } [A(t)] = a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t).$$

Liouville's Theorem Assume $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of the vector differential equation $x' = A(t)x$ on I and Φ is the matrix function with columns $\phi_1, \phi_2, \cdots, \phi_n$. Then, if $t_0 \in I$,

$$\det \Phi(t) = e^{\int_{t_0}^{t} \text{ tr } [A(s)]ds} \det \Phi(t_0),$$

for $t \in I$.

Corollally to Liouville's Theroem Assume $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of the vector differential equation $x' = A(t)x$ on I and Φ is the matrix function with columns $\phi_1, \phi_2, \cdots, \phi_n$. Then either

(a) $\det \Phi(t) = 0$ for all $t \in I$, or

(b) $\det \Phi(t) \neq 0$ for all $t \in I$.

Case (a) holds iff the solutions $\phi_1, \phi_2, cdots, \phi_n$ are linearly dependent on I, while case (b) holds iff the solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I.

Definition An $n \times n$ matrix function Φ is said to be a <u>fundamental matrix</u> for the vector differential equation $x' = A(t)x$ provided Φ is a solution of the matrix equation $X' = A(t)X$ on I and $\det \Phi(t) \neq 0$ on I.

Theorem An $n \times n$ matrix function Φ is a fundamental matrix for the vector differential equation $x' = A(t)x$ iff the columns of Φ are n linearly independent solutions of $x' = A(t)x$ on I. If Φ is a fundamental matrix for the vector differential equation $x' = A(t)x$, then a general solution x of $x' = A(t)x$ is given by

$$x(t) = \Phi(t)c, \quad t \in I,$$

where c is an arbitrary $n \times 1$ constant vector. There are infinitely many fundamental matrices for the differential equation $x' = A(t)x$.