Section 2.1 Introduction to Linear Systems

A linear system of first-order ordinary differential equations is of the form

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t),
\end{aligned}$$

where the functions a_{ij} and $b_i, 1 \le i, j \le n$ are continuous real-valued functions on an interval I. This system can be written as an equivalent vector equation

$$x' = A(t)x + b(t),$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, A(t) := \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \ddots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}, \text{ and } b(t) := \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}$$

for $t \in I$.

We say that an $n \times 1$ vector function x is a solution of the vector equation on I provided x is a continuously differentiable vector function on I and

$$x'(t) = A(t)x(t) + b(t)$$

for all $t \in I$.

Theorem Assume that the $n \times n$ matrix function A and the $n \times 1$ vector function b are continuous on an interval I. Then the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = x_0,$$

where $t_0 \in I$ and x_0 is a given constant $n \times 1$ vector, has a unique solution that exists on the whole interval I.

Definition A family of function $\mathbb{A}$ defined on an interval I is said to be a vector space or linear space provided whenever $x, y \in \mathbb{A}$ it follows that for any constants $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in \mathbb{A}$.

If $\mathbb{A}$ and $\mathbb{B}$ are vector spaces of functions defined on an interval I, then $L : \mathbb{A} \to \mathbb{B}$ is called a linear operator provided

$$L[\alpha x + \beta y] = \alpha L[x] + \beta L[y],$$

for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{A}$.

Since the differential equation $x' = Ax+b$ can be written in the form $Lx = b$, where $L[x](t) := x'(t) - A(t)x(t)$, we call $x' = Ax + b$ a linear vector differential equation. If b is not the trivial vector function, then the equation $Lx = b$ is called a nonhomogeneous linear vector differential equation and $Lx = 0$ is called the corresponding homogeneous linear vector differential equation.