Section 1.3 Autonomous Equations

Definition If $f(x_0) = 0$ we say that x_0 is an equilibrium point for the differential equation $x' = f(x)$.

Note that if x_0 is an equilibrium point for the differential equation $x' = f(x)$, then the constant function $x(t) = x_0$ is a solution of $x' = f(x)$ on $\mathbb{R}$.

Definition Let ϕ be a solution of $x' = f(x)$ with maximal interval of existence (α, ω). Then the set

$$\{\phi(t) : t \in (\alpha, \omega)\}$$

is called an orbit for the differential equation $x' = f(x)$.

Theorem Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then two orbits of $x' = f(x)$ are either disjoint sets or are the same set.

Logistic Growth The logistic law of population growth is

$$N' = rN\left(1 - \frac{N}{K}\right),$$

where N is the number of individuals in the population, $r\left(1 - \frac{N}{K}\right)$ is the per capita growth rate that declines with increasing population, and $K > 0$ is the carrying capacity of the environment.

Definition Assume $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then we let $\phi(\cdot, x_0)$ denote the solution of the IVP

$$x' = f(x), \quad x(0) = x_0.$$

Definition We say that an equilibrium point x_0 of the differential equation $x' = f(x)$ is stable provided given any $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x_1 - x_0| < \delta$ it follows that the solution $\phi(\cdot, x_1)$ exists on $[0, \infty)$ and

$$|\phi(t, x_1) - x_0| < \epsilon,$$

for $t \geq 0$. If, in addition, there is a $\delta_0 > 0$ such that $|x_1 - x_0| < \delta_0$ implies that

$$\lim_{t \to \infty} \phi(t, x_1) = x_0,$$

then we say that the equilibrium point x_0 is asymptotically stable. If an equilibrium point is not stable, then we say it is unstable.