Section 1.1 Basic Results for First-Order Differential Equations

Definition We say that a function x is a *solution* of $x' = f(t,x)$ on an interval $I \subset (a,b)$ provided $c < x(t) < d$ for $t \in I$, x is a continuously differentiable function on I, and

$$x'(t) = f(t, x(t)),$$

for $t \in I$.

Definition Let $(t_0, x_0) \in (a,b) \times (c,d)$ and assume f is continuous on $(a,b) \times (c,d)$. We say that the function x is a solution of the initial value problem (IVP)

$$x' = f(t,x), \quad x(t_0) = x_0,$$

on an interval $I \subset (a,b)$ provided $t_0 \in I$, x is a solution of $x' = f(t,x)$ on I, and

$$x(t_0) = x_0.$$

Theorem Assume $f : (a,b) \times (c,d) \to \mathbb{R}$ is continuous, where $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$. Let $(t_0, x_0) \in (a,b) \times (c,d)$, then the IVP

$$x' = f(t,x), \quad x(t_0) = x_0,$$

has a solution x with a maximal interval of existence $(\alpha, \omega) \subset (a,b)$, where $\alpha < t_0 < \omega$. If $a < \alpha$, then

$$\lim_{t \to \alpha^+} x(t) = c, \text{ or } \lim_{t \to \alpha^+} x(t) = d$$

and if $\omega < b$, then

$$\lim_{t \to \omega^-} x(t) = c, \text{ or } \lim_{t \to \omega^-} x(t) = d.$$

If, in addition, the partial derivative of f with respect to x, f_x, is continuous on $(a,b) \times (c,d)$, then the preceding IVP has a unique solution.