Section 15.2 Line Integrals

If f is defined on a smooth curve C given by

$$C: \; x = x(t), \quad y = y(t), \quad a \le t \le b,$$

then the line integral of f along C is

$$\int_C f(x,y)dS = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

If f is continuous,

$$\int_C f(x,y)dS = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

Two other line integrals are called the line integrals of f along C with respect to x and y:

$$\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

where $C: x = x(t), \; y = y(t), \; a \le t \le b.$

Notation: $\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy.$

Parameterizing a curve C: We can parameterize a line segment that starts at the point (x_0, y_0, z_0) and ends at the point (x_1, y_1, z_1) by

$$C: (x,y,z) = (1-t)(x_0,y_0,z_0) + t(x_1,y_1,z_1), \quad 0 \le tleq1.$$

Note that this gives

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t, \quad z = z_0 + (z_1 - z_0)t, \quad 0 \le t \le 1.$$

In general, a given parametrization $x = x(t), y = y(t), a \le t \le b$, determines an orientation of a curve C, with the positive direction corresponding to increasing values of the parameter t.

Line Integrals of Vector Fields Previously: The work done by a variable force $f(x)$ in moving a particle from a to b along the x-axis is $W = \displaystyle\int_a^b f(x)dx$. The work done by a constant force $\mathbf{F}$ in moving an object from a point P to another point Q in space is $W = \mathbf{F} \cdot \mathbf{D}$.

Now, we can use line integrals to compute the work done by a force field in moving a particle along a smooth curve C :

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

and

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}.$$