Section 13.8 Maxima and Minima of Functions of Two Variables

A function of two variables has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). (This means that $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b).) The number $f(a, b)$ is called a local maximum value. If $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b), then $f(a, b)$ is a local minimum value.

If the inequalities hold for all points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).

Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

A point (a, b) is called a critical point of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or if one of these partial derivatives does not exist.

Second Derivative Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In Case 3, the point (a, b) is called a saddle point of f.

If $D = 0$, the test is inconclusive.

Absolute Maximum and Minimum Values A closed set in $\mathbb{R}^2$ is one that contains all its boundary points.

A bounded set in $\mathbb{R}^2$ is one that is contained within some disk.

Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.