Section 13.6 Directional Derivatives and Gradients

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} =< a, b >$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

provided this limit exists.

Theorem If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} =< a, b >$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) =< f_x(x, y), f_y(x, y) >= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Note: $D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

Maximizing the Directional Derivative Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Tangent Planes to Level Surfaces Suppose S is a surface with equation $F(x, y, z) = k$ and let $P(x_0, y_0, z_0)$ be a point on S. Let C be any curve that lies on S and passes through P. Then we can represent $C = \mathbf{r}(t) =< x(t), y(t), z(t) >$. We assume $\mathbf{r}(t_0) =< x_0, y_0, z_0 >$. With this notation, we have

$$F(x(t), y(t), z(t)) = k.$$

Applying the Chain Rule, we have

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial z} \cdot dzdt = 0.$$

Since $\nabla F =< F_x, F_y, F_z >$ and $\mathbf{r}'(t) =< x'(t), y'(t), z'(t) >$, this can also be written as

$$\nabla F \cdot \mathbf{r}'(t) = 0.$$

In particular, when $t = t_0$, this says

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0.$$

This says that the gradient vector at P is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P. Thus, we define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that contains the point $P(x_0, y_0, z_0)$ and has normal vector $\nabla F(x_0, y_0, z_0)$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$