Section 12.3 Change of Parameter; Arc Length

For the curve

$$\mathbf{r}(t) =< f(t), g(t), h(t) >, \quad a \leq t \leq b,$$

we have the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

If the curve is traversed exactly once as t increases from a to b, then the length of the curve is

$$\begin{aligned} L &= \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt \\ &= \int_a^b |\mathbf{r}'(t)| dt. \end{aligned}$$

If $\mathbf{r}(t) =< f(t), g(t), h(t) >, a \leq t \leq b$, we define its arc length function s by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\frac{dx}{du}^2 + \frac{dy}{du}^2 + \frac{dz}{du}^2}\, du.$$

$s(t)$ represents the length of the curve traced out by $\mathbf{r}(t)$ from $\mathbf{r}(a)$ to $\mathbf{r}(t)$.

By Fundamental Theorem of Calculus, differentiating

$$s(t) = \int_a^t |\mathbf{r}'(u)| du$$

we find

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

It is often useful to parametrize a curve with respect to arc length because arc length does not depend on a particular coordinate system. If a curve $\mathbf{r}(t)$ is given in terms of a parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of $s : t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for $t : \mathbf{r} = \mathbf{r}(t(s))$.