Section 11.4 Cross Products

If $\mathbf{a}$ and $\mathbf{b}$ are nonzero three-dimensional vectors, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}||\mathbf{b}| \sin(\theta))\mathbf{n}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the right-hand rule: If the fingers of your right hand curl through the angle θ from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$.

Notes

1. If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, then we define $\mathbf{a} \times \mathbf{b}$ to be $\mathbf{0}$.
2. Because $\mathbf{a} \times \mathbf{b}$ is a scalar multiple of $\mathbf{n}$, it has the same direction as $\mathbf{n}$ and so $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
3. Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Properties of the Cross Product If $\mathbf{a}, \mathbf{b}$, and $\mathbf{d}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{d}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{d}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{d} = \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{d}$

Theorem The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

The Cross Product in Component Form If $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, then

$$\mathbf{a} \times \mathbf{b} =< a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 > .$$

Equivalently,

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\
&= (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.
\end{aligned}$$

Triple Products The volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$