Section 11.2 Vectors

The term vector is used to indicate a quantity that has both magnitude and direction. (For example, velocity has magnitude and direction; speed only has magnitude.)

The zero vector, denoted by $\mathbf{0}$, has length 0 and is the only vector with no specific direction.

Vector Addition If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u}+\mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Scalar Multiplication If c is a scalar and $\mathbf{v}$ is a vector, then the scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

Notes: Two nonzero vectors are parallel if they are scalar multiples of one another.

The vector $-\mathbf{v} = (-1)\mathbf{v}$ has the same length as $\mathbf{v}$ but points in the opposite direction. $-\mathbf{v}$ is called the negative of $\mathbf{v}$. Thus, the difference $\mathbf{u}-\mathbf{v}$ means $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$.

Components By introducing a coordinate system and setting the initial point of a vector at the origin, we can work with vectors algebraically. The coordinates of the terminal point are called the components of the vector and are denoted by $\mathbf{a} =< a_1, a_2, a_3 >$.

The magnitude or length of the vector $\mathbf{v} =< v_1, v_2, v_3 >$ is denoted by $|\mathbf{v}|$ or $\|\mathbf{v}\|$ and is calculated:

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Theorem If $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$, then

$$\begin{aligned} \mathbf{a}+\mathbf{b} &=< a_1 + b_1, a_2 + b_2, a_3 + b_3 > \\ \mathbf{a}-\mathbf{b} &=< a_1 - b_1, a_2 - b_2, a_3 - b_3 > \\ c\mathbf{a} &=< ca_1, ca_2, ca_3 > . \end{aligned}$$

We denote by V_3 the set of all three-dimensional vectors and, more generally, V_n is the set of all n-dimensional vectors:

$$\mathbf{a} =< a_1, a_2, \cdots, a_n > .$$

Properties of Vectors If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0} = \mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a}+\mathbf{b}) = c\mathbf{a}+c\mathbf{b}$
6. $(c+d)\mathbf{a} = c\mathbf{a}+d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

By defining $\mathbf{i} =< 1, 0, 0 >, \mathbf{j} =< 0, 1, 0 >$, and $\mathbf{k} =< 0, 0, 1 >$, we see that any vector in V_3 can be written as a combination of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$:

$$\begin{aligned} \mathbf{v} =< v_1, v_2, v_3 > &= v_1 < 1, 0, 0 > + v_2 < 0, 1, 0 > + v_3 < 0, 0, 1 > \\ &= v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}. \end{aligned}$$

We call $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ the standard basis for V_3.

A unit vector is a vector whose length is 1. $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ are unit vectors and, in general, if $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$