Section 3.4 Subsequences and the Bolzano-Weierstrass Theorem

If $X = (x_n)$ is a sequence and $n_1 < n_2 < \cdots < n_k < \cdots$ is a strictly increasing sequence of natural numbers, then the sequence $X' = (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots)$ is a subsequence of X.

Theorem If $x_n \to x$, then $x_{n_k} \to x$ for any subsequence (x_{n_k}).

Divergence Criterion If (x_n) satisfies either:

1. $\exists$ two convergent subsequences of (x_n) whose limits are not equal, or
2. (x_n) is unbounded

then (x_n) is divergent.

Monotone Subsequence Theorem For any sequence (x_n) of real numbers, there exists a subsequence (x_{n_k}) that is monotone.

Bolzano-Weierstrass Theorem A bounded sequence of real numbers has a convergent subsequence.

The limit superior of (x_n), denoted $\limsup(x_n)$ or $\overline{\lim}(x_n)$, is given by

$$\begin{aligned}\limsup(x_n) &= \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)\\ &= \inf_{n\in\mathbb{N}}\left(\sup_{m\geq n} x_m\right).\end{aligned}$$

The limit inferior of (x_n), denoted $\liminf(x_n)$ or $\underline{\lim}(x_n)$, is given by

$$\begin{aligned}\liminf(x_n) &= \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)\\ &= \sup_{n\in\mathbb{N}}\left(\inf_{m\geq n} x_m\right).\end{aligned}$$