TEST 3
MATH 310

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To receive full credit on each question, you must show **all** work! This assignment is due at **the beginning of class on Wednesday, December 2nd.** You are allowed to use your class notes, the course textbook, and me as a resource. **No other resources are permitted, including but not limited to: professors or students, other textbooks, the internet, etc.** By signing your name on this test, you pledge to abide by these guidelines. Good luck!

1. Show that the smallest element of a nonempty set of positive integers is unique. (Hint: Use induction.)

2. A sequence $a_0, a_1, a_2, \cdots$ is defined recursively as follows:

$$a_0 = 2;$$
$$a_{n+1} = (a_n)^2.$$

Find a formula for a_n and prove that your formula is correct.

3. Let V be a relation on $\mathbb{R}$ given by $(x, y) \in V$ iff $x = y$ or $xy = 1$. Prove that V is an equivalence relation. Give the equivalence class of 3; of $-\frac{2}{3}$; of 0.

4. Suppose that R is nonempty. Prove that if R is a symmetric, transitive relation on A and the domain of R is A, then R is reflexive on A.

1. We'll use induction for the number of elements in the set.
Base case suppose $A=\{x\}$. Then x is the smallest element (and unique).
IH A set with k elements has a unique smallest element for some $k \in \mathbb{N}$.
Let A represent a set with $k+1$ elements and $x \in A$. Then $A-\{x\}$ has k elements, and therefore has a unique smallest element, call it y.

Case 1 $x<y$: If $x<y$, x is the unique smallest element of A.

Case 2 $x>y$: If $x>y$, y is the unique smallest element of A.

Note $x \neq y$ since we don't write repeated elements in sets.

$\therefore$ by PMI, every subset of $\mathbb{N}$ has a unique smallest element.

2. WTS: $(a_n) = 2^{2^n}$

Base Case $n=1$

$$a_1 = 4 = 2^2$$

IH $\exists k \in \mathbb{N}$ s.t. $a_k = 2^{2^k}$

Then $a_{k+1} = (a_k)^2 = (2^{2^k})^2 = (2^{2^k})(2^{2^k})$

$$= 2^{2^k + 2^k}$$
$$= 2^{2(2^k)}$$
$$= 2^{2^{k+1}}$$

$\therefore$ by PMI, $a_n = 2^{2^n}$ $\forall n \in \mathbb{N}$.

3. Reflexive Let $x \in \mathbb{R}$. Then $x = x$ so xVx.

Symmetric $xVy \Rightarrow x=y$ or $xy=1 \Rightarrow y=x$ or $yx=1 \Rightarrow$ yVx.

Transitive suppose xVy and yVz. Then we have the following cases:

① $x=y$ and $y=z \Rightarrow x=z \Rightarrow xVz$

② xy and $yz=1 \Rightarrow xz=1 \Rightarrow xVz$

③ $xy=1$ and $y=z \Rightarrow xz=1 \Rightarrow xVz$

④ $xy=1$ and $yz=1 \Rightarrow x=z \Rightarrow xVz$

$3/V = \{3, 1/3\}$ $\qquad$ $0/V = \{0\}$

$-2/3 / V = \{-2/3, -3/2\}$

4. Let $x \in A$. Since $A = \text{dom}(R)$, $\exists y \in A$ s.t. $(x,y) \in R$. Since R is symmetric, $(y,x) \in R$. Since R is transitive, $(x,x) \in R$. Thus, R is reflexive.