TEST 3
MATH 310

NAME: Key

November 23, 2015

Each question is worth 10 points. Good luck!

1. Let R be a relation on set A. Prove that $(R^{-1})^{-1} = R$.

Let $(x,y) \in (R^{-1})^{-1}$. Then $(y,x) \in R^{-1} \Rightarrow (x,y) \in R$.
Thus, $(R^{-1})^{-1} \subseteq R$.
Now, let $(x,y) \in R$. Then $(y,x) \in R^{-1}$ and so $(x,y) \in (R^{-1})^{-1}$.
Thus, $R \subseteq (R^{-1})^{-1}$ and so $(R^{-1})^{-1} = R$.

2. Prove that $5^n - 2^n$ is divisible by 3 for every natural number n.

Base case $n=1$

$5^1 - 2^1 = 3 = 3(1)$

IH $\exists k \in \mathbb{N}$ s.t. $5^k - 2^k = 3l$ for some $l \in \mathbb{Z}$.

Then
$$\begin{aligned} 5^{k+1} - 2^{k+1} &= 5(5^k) - 2(2^k) \\ &= 5(3l + 2^k) - 2(2^k) \\ &= 3(5l) + 5(2^k) - 2(2^k) \\ &= 3(5l) + 3(2^k) \\ &= 3(5l + 2^k). \end{aligned}$$

$\therefore$ by PMI, $5^n - 2^n$ is divisible by 3 $\forall n \in \mathbb{N}$.

3. Let $A = \{$ all words in the English language $\}$. Define a relation R on A by xRy if and only if x and y have at least one letter in common. Answer the questions below, justifying each answer with at least a one sentence explanation.

a) Is R reflexive?

Yes since every word has a letter in common with itself.

b) Is R symmetric?

Yes since if word x has a letter in common with word y, then word y has a letter in common with word x.

c) Is R transitive?

NO: $x = dog$
$y = log$
$z = late$

xRy and yRz but x and z share no common letters.

4. Let $A = \{a, b, c, d\}$. Give an example of relations R, S, and T on A such that $S \circ R = T \circ R$ but $S \neq T$.

$R = \{(a,b)\}$

$S = \{(b,c), (a,d)\}$

$T = \{(b,c), (a,b)\}$

5. Let T be a relation on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)T(a, b)$ iff $x^2 + y^2 = a^2 + b^2$. Prove that T is an equivalence relation. Sketch the equivalence class of $(1, 2)$; of $(4, 0)$.

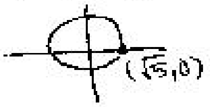
Reflexive $\forall (x,y) \in \mathbb{R}\times\mathbb{R},\ x^2+y^2 = x^2+y^2$.

Symmetric $x^2+y^2 = a^2+b^2 \Rightarrow a^2+b^2 = x^2+y^2$.

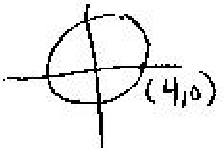
Transitive $x^2+y^2 = a^2+b^2$ and $a^2+b^2 = u^2+v^2 \Rightarrow$

$x^2+y^2 = u^2+v^2$.

$(1,2)/_T = \{(x,y) : x^2+y^2 = 5\}$



$(4,0)/_T = \{(x,y) : x^2+y^2 = 16\}$



6. Prove, using the Principle of Mathematical Induction, that

$$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \quad \forall n \in \mathbb{N}.$$

Base case $n=1$

$$\sum_{i=1}^{1} 2^i = 2 = 2^2 - 2$$

IH $\exists k \in \mathbb{N}$ s.t. $\sum_{i=1}^{k} 2^i = 2^{k+1} - 2$

Then $\sum_{i=1}^{k+1} 2^i = \sum_{i=1}^{k} 2^i + 2^{k+1} = (2^{k+1} - 2) + 2^{k+1}$

$= 2(2^{k+1}) - 2$

$= 2^{k+2} - 2$.

$\therefore$ by PMI, $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2 \quad \forall n \in \mathbb{N}$.