TEST 2
MATH 310

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You may only use the course textbook, the instructor lecture notes, and your class notes. You are not allowed to work together. You are also not allowed to ask anyone questions other than Dr. Kristen Abernathy. By signing your name above, you are pledging the university honor code: "I have neither received unauthorized aid nor given aid on this assignment." Each question is worth 10 points. This is due at the beginning of class on Wednesday, October 28th. Good luck!

1. Let A and B be sets. Prove $A \subseteq B$ iff $A - B = \varnothing$.

2. Prove that there do not exist integers m and n such that $12m + 15n = 1$.

3. Let a and b be real numbers. Prove that $|a + b| \leq |a| + |b|$.

4. Prove or disprove: If a and b are positive integers, a divides b, and b divides a, then $a = b$.

1. We'll first prove $A \subseteq B$ implies $A - B = \varnothing$.
Suppose $A \subseteq B$ and $A - B \neq \varnothing$. Let $x \in A - B$. Then $x \in A$ and $x \notin B$. But, since $A \subseteq B$, $x \in A \rightarrow x \in B$ ↯.
Thus, if $A \subseteq B$, $A - B = \varnothing$.
Now, we'll prove $A - B = \varnothing$ implies $A \subseteq B$.
Suppose $A - B = \varnothing$ and let $x \in A$. Since $x \notin A - B$, $x \notin A$ or $x \in B$. Since $x \in A$, it must be that $x \in B$.
Thus, $A \subseteq B$.

2. Suppose $\exists\, m, n \in \mathbb{Z}$ s.t. $12m + 15n = 1$. Then $3(4m + 5n) = 1$, where $4m + 5n \in \mathbb{Z}$.

This implies $3 \mid 1$ ↯.

3. Let $a, b \in \mathbb{R}$ and recall $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Case 1 $a \geq 0$ and $b \geq 0$:

If $a \geq 0$ and $b \geq 0$, $a+b \geq 0$ and so

$$|a+b| = a+b = |a| + |b|.$$

Case 2 $a < 0$ and $b < 0$:

If $a<0$ and $b<0$, $a+b<0$ so

$$|a+b| = -(a+b) = -a-b = |a|+|b|.$$

Case 3 WLOG, $a \geq 0$ and $b<0$

Subcase 1 $|a| \geq |b|$

Since $|a| \geq |b|$, $a+b \geq 0$ so

$$|a+b| = a+b < a-b = |a|+|b|.$$

Subcase 2 $|a| < |b|$

Since $|a| < |b|$, $a+b<0$ so

$$|a+b| = -(a+b) = -a-b < a-b = |a|+|b|.$$

Thus, in all possible cases, $|a+b| \leq |a|+|b|$.

4. Suppose $a|b$ and $b|a$. Then $\exists\, k, \ell \in \mathbb{Z}$ s.t.

$$b = ak \text{ and } a = b\ell \rightarrow b = ak = (b\ell)k.$$

This gives $\ell k = 1$. Since $a>0$ and $b>0$, $\ell>0$ and $k>0$ and so, the only possibility is that $\ell = 1$ and $k=1$. Thus, $a=b$.