Section 6.2 More Examples of Mathematical Induction

Claim $5^n < n!$ for $n \geq 12$.

Use statements from the following table, along with the symbols $<$ and $=$ to construct a proof of the above claim:

5^{12}	$\exists k \geq 12$ such that
$12!$	$5(5^k)$
$(k+1)!$	5^k
479,001,600	$5(k!)$
5^{k+1}	$244, 140, 625$
$(k+1)k!$	$k!$

Claim 2 divides $n^2 + n$ for all $n \in \mathbb{N}$.

Use statements from the following table, along with the symbol $=$ and logical connectives to construct a proof of the above claim:

$k^2 + 2k + 1 + k + 1$	$\exists k \in \mathbb{N}$ such that
$1^2 + 1 = 2$	$k^2 + k = 2l$
$\exists l \in \mathbb{Z}$ such that	$k^2 + k + 2k + 2$
$2(l + k + 1)$	$2 = 2(1)$
2 divides $k^2 + k$	$(k+1)^2 + (k+1)$
$2l + 2k + 2$	2 divides $(k+1)^2 + (k+1)$

What's wrong with the following proof that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for all n? If the claim is true, correct the proof; if it is false, provide a counterexample.

Proof

Base Case n = 1:

$$\begin{aligned} 1 &= \frac{1(1+1)}{2} \\ 1 &= \frac{2}{2} \\ 1 &= 1 \ \checkmark \end{aligned}$$

Inductive Hypothesis Suppose there exists $k \in \mathbb{N}$ such that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

WTS:

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= \frac{(k+1)((k+1)+1}{2} \\ \frac{k(k+1)}{2} + k + 1 &= \frac{(k+1)(k+2)}{2} \\ \frac{k^2 + k + 2k + 2}{2} &= \frac{k^2 + 3k + 2}{2} \\ \frac{k^2 + 3k + 2}{2} &= \frac{k^2 + 3k + 2}{2} \ \checkmark \end{aligned}$$

What's wrong with the following proof that

$$3 \cdot 3^1 + 5 \cdot 3^2 + \cdots + (2n+1) \cdot 3^n = n \cdot 3^{n+1}$$

for all $n \in \mathbb{N}$? If the claim is true, correct the proof; if it is false, provide a counterexample.

Proof We use mathematical induction. Let k be an arbitrary natural number, and suppose that

$$3 \cdot 3^1 + 5 \cdot 3^2 + \cdots (2k+1) \cdot 3^k = k \cdot 3^{k+1}.$$

Then

$$\begin{aligned} 3 \cdot 3^1 + 5 \cdot 3^2 + \cdots (2k+1) \cdot 3^k + (2k+3) \cdot 3^{k+1} &= k \cdot 3^{k+1} + (2k+3) \cdot 3^{k+1} \\ &= (3k+3) \cdot 3^{k+1} \\ &= (k+1)3^{k+2}. \end{aligned}$$

Try it on your own! Prove the following claims using mathematical induction:

Claim 6 divides $n^3 - n$ for all $n \in \mathbb{N}$.

Claim If A is a set with n elements, then A has $\frac{n(n-1)}{2}$ 2-element subsets.