Math 305

Sections 4.1 & 4.3 Second-Order, Linear, Homogeneous Equations

Definition A second-order differential equation is of the form

$$y'' = F(t, y, y').$$

A linear second-order equation has the special form

$$y'' + p(t)y' + q(t)y = g(t).$$

A homogeneous linear equation is of the form

$$y'' + p(t)y' + q(t)y = 0.$$

Proposition Suppose y_1 and y_2 are solutions to the homogeneous linear equation

$$y'' + p(t)y' + q(t)y = 0,$$

then $y = c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

Note: $y = c_1y_1 + c_2y_2$ is called a linear combination of y_1 and y_2.

Definition Two functions u and v are linearly independent on the interval (α, β) if neither is a constant multiple of the other on (α, β). If one is a constant multiple of the other on (α, β), then u and v are linearly dependent on (α, β).

Theorem Suppose y_1 and y_2 are linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$. Then the general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

where c_1 and c_2 are arbitrary constants.

Proposition If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two distinct real roots λ_1 and λ_2, then the general solution to $y'' + py' + q = 0$ is

$$y(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t},$$

where $c_1, c_2 \in \mathbb{R}$.

Theorem Suppose $p(t)$, $q(t)$, $g(t)$ are continuous on (α, β). Let $t_0 \in (\alpha, \beta)$. Then for any real numbers y_0, y_1, there is a unique function $y(t)$ defined on (α, β) which is a solution to

$$y'' + p(t)y' + q(t)y = g(t) \text{ for } \alpha < t < \beta$$

and satisfies the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_1$.

Proposition If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has two complex conjugate roots $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, then the general solution may be written as

$$y(t) = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt).$$

Proposition If the characteristic equation $\lambda^2 + p\lambda + q = 0$ has only one double root $\lambda = -\frac{p}{2}$, then the general solution to $y'' + py' + qy = 0$ is

$$y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \;\; c_1, c_2 \in \mathbb{R}.$$

Definition The <u>Wronskian</u> of two functions u and v is defined to be

$$W(t) = \det \begin{bmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{bmatrix} = u(t)v'(t) - u'(t)v(t).$$

Proposition Suppose u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (α, β). The Wronskian of u and v is either identically 0 on (α, β) or never equal to 0 on (α, β).

Proposition Suppose u and v are solutions to the linear, homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

on the interval (α, β). Then u and v are linearly dependent if and only if their Wronskian is identically 0 on (α, β).