Section 8.2 Separable Equations

Separable Equations

A separable equation is a first-order differential equation which can be written in the form

$$\frac{dy}{dx} = g(x)f(y).$$

If $f(y) \neq 0$, this equation can be separated as follows:

$$\frac{1}{f(y)}dy = g(x)dx.$$

Now, to solve the differential equation, integrate both sides and solve for y :

$$\int \frac{1}{f(y)}dy = \int g(x)dx.$$

Exponential Growth and Decay

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant. This equation is sometimes called the law of natural growth (if $k > 0$) or the law of natural decay (if $k < 0$). This is a separable equation, so we can solve as follows:

$$\int \frac{1}{y}dy = \int kdt \implies$$

$$ln|y| = kt + C \implies$$

$$y = Ae^{kt}.$$

Notice that $y(0) = A$, so A is the initial value of the function.

Population Growth

Notice that

$$\frac{dP}{dt} = kP \implies k = \frac{1}{P}\frac{dP}{dt}.$$

In this case, k is called the relative growth rate.

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiation. It has been found experimentally that radioactive substances decay at a rate proportional to the remaining mass. That is,

$$\frac{dm}{dt} = km \ \ (k < 0)$$

or, by solving this separable equation,

$$m(t) = m_0e^{kt}.$$

Physicists express the rate of decay in terms of half-life, the time required for half of any quantity to decay.